The $H^\infty$ Optimal Control of Linear Systems with Actuators Magnitude and Rate Saturation

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Abstract

The proposed controller design algorithm in this paper can take into account independently both the magnitude and rate constraints on actuator dynamics. The control law is constructed in terms of linear matrix inequalities. A suitable chosen initial state condition, which can represent the desired set of tracking commands, is treated as a parameter for the controller design algorithm. By the example of satellite attitude control, the effectiveness of the proposed algorithm is verified through the achievable $H^\infty$ performance and the time response simulations with specified constraints on the magnitude and rate of the control torque.

1. Introduction

The magnitude and rate saturation on actuator dynamics are prevailing in physical control systems and substantially affect the behavior of the controlled systems. Therefore, the issues of the magnitude and rate constraints on the actuator saturation must be taken care well during controller design, otherwise, the designated performance can be deteriorated and even the system stability can possibly be destroyed.

Among the works that have been done for systems with both magnitude and rate constraints, [1] are relied on nonlinear control techniques. The works in [2] addressed the magnitude and rate saturation as an anti-windup problem. A design approach with low gain controller with respect to the actuator magnitude and rate saturation was proposed in [3]. The approaches in [4] synthesized the magnitude and rate saturation compensators using the frequency domain techniques. The works in [5] used the convex computation methods, such as the linear matrix inequalities (LMIs), to handle the control system designs with actuator magnitude and rate saturation.

This paper considers the controller design for linear systems with actuator magnitude and rate constraints by augmented diagonal integral dynamics to the systems actuator. The inputs of the integrator present the rate of the control efforts. The augmented dynamics formulate the systems control efforts as additional states. A state feedback control law is expressed in terms of LMIs to achieve optimization of $H^\infty$ performance while subject to the specified constraints on the magnitude and rate of the actuators. The suitable set of initial states representing desired ranges of tracking commands is assumed and addressed in the LMI algorithm. The feedback of the new state, the original control efforts, in the design scheme forms an inner loop of the controlled systems and represents the suitable actuator dynamics by the chosen feedback gain such that the desired $H^\infty$ performance can be optimized and the magnitude the rate constraints can be satisfied.

2. $H^\infty$ Optimal Control via LMIs

PROBLEM STATEMENT Consider the linear system

\begin{align}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= Cx + D_1 w + D_2 u
\end{align}

with regulated output,

\begin{align}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= Cx + D_1 w + D_2 u
\end{align}

The design objective is to achieve a certain level of convergence rate $\beta > 0$ and a minimized level of $H^\infty$ performance $\gamma$ from disturbance $w$ to regulated output $z$ while satisfy the actuator magnitude and rate constraints,

\begin{align}
|u| &\leq u_{\text{max}} \\
|\dot{u}| &\leq \dot{u}_{\text{max}}
\end{align}

for given initial condition $x_0$.

In order to address the actuator rate saturation, let the derivative of the control effort denoted as, $\dot{u} = v$. 
Considering the control effort as augmented state, the linear dynamic systems and the regulated output are written as,

\[
\begin{aligned}
\dot{\mathbf{x}} &= (A \quad B_2) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} + (B_1 \quad 0) \mathbf{w} + (0 \quad 1) \mathbf{v} \\
\mathbf{z} &= (C_1 \quad D_1) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} + D_2 \mathbf{w}
\end{aligned}
\] (3a)

It is noted that the control effort \( \mathbf{u} \) has been treated as part of the state variables, while the derivative of the control effort \( \mathbf{v} \) is formulated as the new control effort to be manipulated in the augmented dynamics. With the denotation,

\[
\begin{aligned}
\mathbf{x}_u &= \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \\
\mathbf{v}_u &= \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}, \\
\mathbf{w}_u &= \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix}
\end{aligned}
\]

and the state feedback control law,

\[
\mathbf{v} = -(F \quad F_c) \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} := -\tilde{F} \mathbf{x},
\] (5)

with \( F \in \mathbb{R}^{n \times n}, F_c \in \mathbb{R}^{m \times m} \), where \( n \) is the dimension of the state \( \mathbf{x} \) and \( m \) is the dimension of the control effort \( \mathbf{u} \).

The closed-loop controlled systems can be written as,

\[
\begin{aligned}
\dot{\mathbf{x}} &= \tilde{A} \mathbf{z} + \tilde{B} \mathbf{w} \\
\mathbf{z} &= \tilde{C} \mathbf{z} + D \mathbf{w}
\end{aligned}
\] (6)

with \( \tilde{A} = A - \tilde{B} \tilde{F} \).

According to the Bounded-Real Lemma [6], the \( \mathcal{H}^\infty \) performance \( \|T_{sr}\| < \gamma \) can be satisfied if the following matrix inequality holds under the positive Lyapunov matrix \( \mathbf{P} \),

\[
\begin{pmatrix}
\tilde{A} \mathbf{P} + \mathbf{P} \tilde{A}^T & \mathbf{P} \tilde{B}_1 & \mathbf{C}_1^T \\
\tilde{B}_1^T \mathbf{P} & -\gamma \mathbf{I} & \mathbf{D}_1^T \\
\mathbf{C}_1 & \mathbf{D}_1 & -\gamma \mathbf{I}
\end{pmatrix} < 0
\] (7)

By denoting \( \mathbf{P}^T := \mathbf{Q} \), then multiplying the congruence matrices, diag\( (\mathbf{Q}, \mathbf{I}, \mathbf{I}) \), from left and right side of the block matrix, and let \( \mathbf{L} := \tilde{F} \mathbf{Q} \), the matrix inequality can be written as

\[
\begin{pmatrix}
\mathbf{Q} \tilde{A}^T + \tilde{A} \mathbf{Q} + \tilde{A} \tilde{B}_1 + \mathbf{B}_2 \mathbf{L} & \mathbf{B}_1 & \mathbf{Q} \mathbf{C}_1^T \\
\tilde{B}_1^T \mathbf{Q} & -\gamma \mathbf{I} & \mathbf{D}_1^T \\
\mathbf{C}_1 & \mathbf{D}_1 & -\gamma \mathbf{I}
\end{pmatrix} < 0
\] (8)

which is already in a LMI formulation with matrix variables \( \mathbf{Q} > 0 \) and \( \mathbf{L} \). Once the matrix variables \( \mathbf{Q} \) and \( \mathbf{L} \) are solved, the state feedback law can be computed as \( \tilde{F} = \mathbf{L} \mathbf{Q}^{-1} \).

If the matrix inequality can be established for system matrix \( A \) replaced by \( A_p = A + \beta \mathbf{I} \) in \( \tilde{A} \), then the close-loop controlled system will achieve a convergence rate \( \beta > 0 \).

Assumed the initial condition is given as \( \mathbf{x}_0 \), which may represent desired tracking command for the controlled system. Let the Lyapunov matrix \( \mathbf{P} \) satisfy,

\[
\mathbf{v}_0^T \mathbf{P} \mathbf{x}_0 < 1,
\]

then it can be written as the following LMI in terms of the synthesis matrix variable \( \mathbf{Q} = \mathbf{P}^{-1} > 0 \),

\[
\begin{pmatrix}
1 \\
\mathbf{x}_0^T \\
\mathbf{x}_0^T \\
\mathbf{Q}
\end{pmatrix} > 0
\] (9)

If the controlled closed-loop system is stabilized, then we have the condition on the state \( \mathbf{x} \),

\[
\mathbf{x}^T \mathbf{P} \mathbf{x} \leq \mathbf{x}_0^T \mathbf{P} \mathbf{x}_0 < 1,
\] (10)

The magnitude constraints on the control efforts of the augmented dynamics \( \mathbf{v} \), that is, the rate constraints on the original control effort \( \mathbf{u} \), is known as \( |\mathbf{v}_i| = ||\mathbf{u}_i||_{\text{max}} \) from (2), which is equivalent to,

\[
\mathbf{x}^T \mathbf{P} \mathbf{x} \leq \mathbf{x}_0^T \mathbf{P} \mathbf{x}_0 < 1,
\]

By (10), the rate constraints in (11) can be established if the following condition is true,

\[
\frac{1}{\mathbf{u}_{\text{max}}} \mathbf{x}^T \mathbf{Q} \mathbf{L}_i \mathbf{Q}^{-1} \mathbf{x} \leq \mathbf{x}_0^T \mathbf{Q}^{-1} \mathbf{x},
\] (12)

which is equivalent to,

\[
\frac{1}{\mathbf{u}_{\text{max}}} \mathbf{L}_i^T \mathbf{L}_i \leq \mathbf{Q},
\] (13)

and can be written as the following LMI in the synthesis variables \( \mathbf{Q} \) and \( \mathbf{L} \),

\[
\begin{pmatrix}
\mathbf{Q} & \mathbf{L}_i \\
\mathbf{L}_i & \mathbf{u}_{\text{max}}
\end{pmatrix} \geq 0,
\] (14)

where \( \mathbf{L}_i \) denoted the i-th row of \( \mathbf{L} \).

The control effort of the original dynamics is reformulated as the augmented states, that is,

\[
\mathbf{u}_i = \ell_{ir} \mathbf{x}, \quad i = 1, 2, 3, \ldots, m.
\] (15)

As stated in (2), the given magnitude constraint on i-th original control effort, \( \mathbf{u}_i \leq \mathbf{u}_{\text{max}} \), is equivalent to,

\[
\mathbf{x}^T \ell_{ir}^T \ell_{ir} \mathbf{x} \leq \mathbf{u}_{\text{max}}^2.
\] (16)

According to (10), the magnitude constraint in (16) can be established by the following,

\[
\frac{1}{\mathbf{u}_{\text{max}}} \mathbf{x}^T \ell_{ir}^T \ell_{ir} \mathbf{x} \leq \mathbf{Q},
\] (17)

which can be written as,

\[
\frac{1}{\mathbf{u}_{\text{max}}} \mathbf{Q} \ell_{ir}^T \ell_{ir} \mathbf{Q} \leq \mathbf{Q},
\] (18)

and expressed as the following LMI in terms of the synthesis variable \( \mathbf{Q} \),
\[
\begin{bmatrix}
Q & Q_{\text{ref}}^T \\
J_{\text{ref}} & Q
\end{bmatrix} \succeq 0.
\] (19)

PROPOSITION 1 Consider the linear system with disturbance input \( w \) and regulated output \( z \) defined in (1a) and (1b), and given initial condition \( x_0 \). The state-feedback controlled system can achieve a \( \mathcal{H}^\infty \) performance level \( \| T_{zw} \| < \gamma^* \) with convergence rate \( \beta^* > 0 \) if there exist synthesis matrix variables \( Q \) and \( L \) such that the LMI (8) can be established for \( \gamma^* > 0 \) with system matrix \( A \) replaced by \( A_p = A + B_1 L \) in \( \hat{A} \). Moreover, the magnitude and rate constraints for control effort can be satisfied for the given initial condition \( x_0 \), if the LMIs in (9), (14), and (19) can be established for given constraints \( u_{\text{max}} \), \( \dot{u}_{\text{max}} \), and the augmented initial state condition \( \dot{x}_0 \).

3. Numerical Example

The proposed controller design algorithm is demonstrated by the satellite attitude control problem. The satellite model is considered to be two rigid bodies connected by a flexible link. The rigid bodies consist of a main body and an instrumentation module represented by inertias \( J_1 \) and \( J_2 \), respectively. The flexible link is modeled as a spring with torque constant \( k \) and viscous damping \( f \). The dynamic equation is

\[
\begin{bmatrix}
J_{1} & 0 & f (\hat{\theta}_1 - \hat{\theta}_2) + k(\theta_1 - \theta_2) = T + w \\\nJ_{2} & 0 & f (\hat{\theta}_2 - \hat{\theta}_1) + k(\theta_2 - \theta_1) = 0
\end{bmatrix},
\] (20)

where \( \theta_1 \) and \( \theta_2 \) are the attitude angles for the main body and the instrumentation module, \( T \) is the control torque, and \( w \) is the torque disturbance acting on the main body. The control objective is to minimize the \( \mathcal{H}^\infty \) performance level from \( w \) to \( \theta \), and the closed-loop system preserves certain level of convergence rate. The derived state-space representation in the form of the generalized system (1) is written as

\[
\begin{bmatrix}
\dot{x} = Ax + Bu + B_2 w \\
z = Cx,
\end{bmatrix}
\] (21)

where \( x = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)^T \), \( u = T \), and

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
k / J_1 & k / J_1 & -f / J_1 & f / J_1 \\
k / J_2 & -k / J_2 & f / J_2 & -f / J_2
\end{bmatrix},
\] (22a)

\[
B_1 = \begin{bmatrix}
0 \\
0 \\
1 / J_1 \\
0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 \\
0 \\
1 / J_1 \\
0
\end{bmatrix}, \quad C = (0 \ 1 \ 0 \ 0). \] (22b)

The satellite parameter values are \( J_1 = 1, J_2 = 0.1, k = 0.2450, f = 0.0188 \). The state initial conditions considered are \( \dot{x}_0, \text{cmd} = (1, 1, 0, 0, 0, 0) \) and \( \dot{x}_0, \text{cmd} = (-1, -1, 0, 0, 0) \) for representing the satellite normalized tracking commands \( \theta_1, \text{cmd} = \theta_2, \text{cmd} = \pm 1 \). Under the actuators magnitude and rate constraints, \( u_{\text{max}} = \{10^{-2}, 1, 10\} \) and \( \dot{u}_{\text{max}} = \{10^{-2}, 10^{-1}, 1, 10, 10^{-2}\} \), respectively, and the designated convergence rate \( \beta^* = 0.05 \), the obtained optimized \( \mathcal{H}^\infty \) performance levels by Proposition 1 are depicted in Figure 1.

Two designs are adopted for demonstration in the time response simulations for the proposed algorithm. Both of them are constructed under magnitude constraint \( u_{\text{max}} = 1 \). One design with rate constraint \( \dot{u}_{\text{max}} = 10^{-2} \), the achieved \( \mathcal{H}^\infty \) performance level is \( \gamma^* = 69.0450 \) and the state-feedback control law is,

\[
\dot{F}^*_{1,1} = \begin{bmatrix}
0.0011 & 0.0070 & 0.0359 & 0.0352 & 0.2236
\end{bmatrix}
\] (23a)

Another design with rate constraint \( \dot{u}_{\text{max}} = 1 \), the achieved \( \mathcal{H}^\infty \) performance level is \( \gamma^* = 5.8035 \) and the state-feedback control law is,

\[
\dot{F}^*_{1,1} = [0.3076 \ 0.1144 \ 1.1068 \ 0.6283 \ 0.5259]
\] (23b)

The time responses are conducted under normalized tracking commands \( \theta_1, \text{cmd} = \theta_2, \text{cmd} = 1 \) from the origin and the band-limited white noise \( w(t) \). Figures 2 and 3 show the time responses for the controlled systems with control law \( \dot{F}^*_{1,1} \). Figure 2 shows the time responses with the disturbance input imposed. The consistent unsteady ripples are found in the time response of \( \theta_1, \text{cmd} \) but the overall time response is still satisfactory. In both the simulations in Figure 2, the designated constraints on magnitude and rate saturation are well met.

Figure 3 shows the time response that the rate constraint is decreased further to the more restricted \( \dot{u}_{\text{max}} = 10^{-2} \). As shown, the time response is restrained by the over-bounded rate constraint and the system stability can not be maintained even without the disturbance input \( w(t) \). It is noted that the vertical scale has been modified as from –10 to 10.
Figure 4 shows the time responses for the controlled systems with control law $F_{j=0}^*$ when the disturbance input is imposed. Although the ripples can be observed in the time response of $\theta_j(t)$, the rate constraint $\dot{u}_{\text{max}} = 10^{-2}$ has not been reached in all the time and the overall time response is still acceptable.

4. Conclusion

The proposed controller design algorithm in this paper can take into account both the magnitude and rate constraints on actuator dynamics. The control law is constructed based on the initial state conditions. It is demonstrated that the allowable change rate must be well handled; otherwise, not only the designated performance level can not be fulfilled, but also even the stability can not be preserved.

5. References


