Multi-objective controller design for bi-linear stochastic systems via sliding mode control concept

K-Y Chang\textsuperscript{1*}, C-Y Lu\textsuperscript{2}, C-H Shih\textsuperscript{3}, and P-C Chen\textsuperscript{4}

\textsuperscript{1}Department of Electronic Engineering, Chienkuo Technology University, Changhua City, Taiwan, Republic of China
\textsuperscript{2}Department of Industrial Education and Technology, National Changhua University of Education, Changhua City, Taiwan, Republic of China
\textsuperscript{3}Institute of Mechatronoptic Systems, Chienkuo Technology University, Changhua City, Taiwan, Republic of China
\textsuperscript{4}Department of Electrical Engineering, Kao Yuan University, Kaohsiung, Taiwan, Republic of China

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Abstract: A sliding mode controller $u(t)$ is designed to achieve the following three objectives simultaneously: eigenvalue placement, $H_\infty$ norm constraint, and individual state variance constraint for bi-linear stochastic systems. By using the invariance property of sliding mode control, the matched bi-linear term of the system disappears on the sliding mode. With the aid of the upper bound covariance control theory, the controller $u(t)$ is derived in which the control feedback gain matrix $G$ is synthesized for achieving the above multiple objectives.

Keywords: bi-linear stochastic systems, $H_\infty$ norm constraint, eigenvalue placement constraint, sliding mode control, upper bound covariance control

1 INTRODUCTION

Bi-linear stochastic systems occur in real control systems: the stochastic input process may model disturbances, input measurement errors, or a feedback control from a noise output. Some types of these systems, called ‘state-dependent noise (SDN) systems’ occur very often in aerospace systems [1]. Others may be found in uncertain flexible systems [2] and so on. However, when the real systems are modelled as the form of bi-linear stochastic systems, the control designs for various objectives are important topics to be addressed. The covariance control for bi-linear stochastic systems has therefore been widely discussed in literature [3–5]. Chung and Chang [3] developed a methodology of constrained variance design for bi-linear stochastic continuous system. The methodology of choosing a compatible covariance matrix, which is based on the state covariance assignment (SCA) theory, may satisfy the performance object. In reference [4], the authors extended the theory of covariance control for linear discrete time systems to a general class of bi-linear stochastic discrete time systems. Yasuda et al. [5] extended the theory of covariance control to continuous time bi-linear systems. However, the system may become unstable when large parameter uncertainties exist.

In general, it is always necessary to develop ways to design controllers to achieve control systems with multiple objectives. Because certain control objectives, e.g. robust stability and noise attenuation, can be achieved if certain $H_\infty$ bounds are satisfied [6], there have been many papers concerning feedback controller design with $H_\infty$ norm and variance constraints [7, 8]. Other researchers [9, 10] discussed the $H_\infty$ norm and variance constrained problem simultaneously. However, the Riccati equation approach applied by Bernstein et al. [9] and Yeh et al. [10], which minimizes a scalar cost index, does not ensure the satisfaction of the individual variance constraints. Therefore, a more direct methodology named covariance control for controller design to achieve variance constraints of individual states is developed [11]. Since the presence of possible

*Corresponding author: Department of Electronic Engineering, Chienkuo Technology University, No. 1, Chieh Shou North Road, Changhua City 500, Taiwan, Republic of China. email: kychang@cc.ctu.edu.tw
system perturbations is not considered in reference [11], the system may become unstable when it suffers from perturbations. An improved control method, for satisfying variance constraints with perturbations, referred to here as upper bound covariance control (UBCC), was proposed in references [12] and [13]. Nevertheless, the drawback of the direct UBCC approach is that the state feedback gain designed in references [12] and [13] will become very large when the systems suffer from large perturbations. It is well known that a feedback gain is too large to encourage in practical application.

The methodology of conventional eigenvalue assignment controller design has been widely used in control engineering. Considering the characteristics of a closed-loop system, such as its stability, damping ratio, sensitivity, etc., its eigenvalues are not necessarily specified at exact locations, but it is sufficient for them to be assigned in a chosen region. Therefore, some researchers (see references [14] and [15]) have investigated the problem of region eigenvalue placement. For the regional eigenvalue constraint, a typical rule of evaluating the relative stability of closed-loop systems is to judge whether all of the eigenvalues are located within a prescribed circular region. This prescribed circular region is denoted by $D(q, \rho)$ with the centre at $-q+0$ ($q>0$) and the radius $\rho$, ($\rho < q$). This constraint is one of the most frequently employed performance requirements in system control design problems.

Owing to the simple design, easy implementation, and insensitivity to system perturbations, sliding mode control (SMC) has become a successful synthesis method and has been applied to many complex systems. The main advantage of the SMC system is that the system dynamics in the sliding mode are invariant if parameter uncertainties and/or perturbations satisfy a certain matching condition. However, the SMC for stochastic systems has been receiving relatively little attention until recently [16–18]. In particular, the problems that consider SMC for bi-linear stochastic systems are rarely found in the literature [19, 20]. Based on the concept of SMC design, Chang and Chang [16], and Chang and Wang [17, 18] addressed the covariance control problems in the case of stochastic model reference systems, stochastic linear perturbed systems, and stochastic large-scale systems respectively. Moreover, the above approach was successfully extended to bi-linear stochastic systems [19, 20]. In reference [19], using the invariance property of SMC to ensure the nullity of the matched bi-linear term in the system on the sliding mode, the controller can be designed to force the feedback gain matrix to achieve the goal of steady state covariance assignment. In reference [20], as an extension of the results in reference [19], a control is designed to achieve $H_\infty$ norm constraint and individual variance constraint simultaneously. Furthermore, based on the UBCC approach and with specified eigenvalue placement consideration, this paper extends the results in references [19] and [20] to achieve the eigenvalue placement constraint, $H_\infty$ norm constraint, and individual variance constraint for bi-linear stochastic systems. Then, the designed controlled system will have the capability of a fast and good response, noise attenuation, and robust stability. Here, the current authors would like to point out that, according to recently published related papers, some authors [21, 22] are interested in following their approach [16, 17] to deal with the problems of uncertain stochastic systems with time-varying delay.

The remainder of this paper is organized as follows. Section 2 describes the system structure and formulates the problems. In section 3, the sliding phase and hitting phase of the system are studied and a controller $u(t)$ via SMC methodology is also designed. In section 4, the necessity and sufficiency conditions for the existence of the control feedback gain $G$ for the system to achieve the goals are derived. A numerical example is given to demonstrate the control effect of the proposed method in section 5. Finally, some conclusions are presented in section 6.

2 SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a time-invariant bi-linear stochastic multivariable system established on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ and the system is described as

$$\dot{x}(t) = A + \sum_{i=1}^{p} N_i v_i(t)x(t) + Bu(t) + Dw(t) \tag{1a}$$

$$y(t) = Fx(t) \tag{1b}$$

where $A$ and $N_i \in \mathbb{R}^{n \times n}$, $B$ and $D \in \mathbb{R}^{n \times m}$, $F \in \mathbb{R}^{m \times n}$; $x(t) \in \mathbb{R}^{n \times 1}$ and $v_i(t) \in \mathbb{R}^{1 \times 1}$; $u(t)$, $y(t)$ and $w(t) \in \mathbb{R}^{m \times 1}$. Here, it is assumed that the $w(t)$ and $v_i(t)$ are zero-mean mutually independent white noise satisfying equations (2) and (3), respectively.
\[ E(w(t)) = 0, E(x(0)w^T(t)) = 0, E(w(t)v_i(t)) = 0, E(w(t)w^T(t)) = W = I \] (2)

\[ E(v_i(t)) = 0, E(x(0)v_i(t)) = 0, E(w(t)v_i(t)) = 0, E(v_i(t)v_i^T(t)) = V = 1 \] (3)

where \( x(0) \) and \( I \) denote the initial state of \( x(t) \) and the identity matrix, respectively. Suppose that \((A, B)\) is a stabilizable pair, and the matrices \( B \) and \( \sum_{i=1}^{P} N_i v_i(t) \) satisfy the perfect matching condition, i.e.

\[
\text{rank } \left[ B : \sum_{i=1}^{P} N_i v_i(t) \right] = \text{rank } [B] \tag{4}
\]

Moreover, it is also assumed that the range space of matrix \( B \) intercepts the range space of matrix \( D \) only at the origin.

**Remark 1**

The matching condition (4) can also be expressed as

\[
\text{rank } \left[ B : \sum_{i=1}^{P} N_i \right] = \text{rank } [B]\]

for \( v_i(t) \in \mathbb{R}^{1 \times 1} \).

The goals of this paper are as follows: how to design the controller \( u(t) \) for the system such that:

(a) the eigenvalues of closed-loop system are located within the specified disk \( \mathcal{D}(q, \rho) \), i.e. \( \lambda(A + BG) \in \mathcal{D}(q, \rho) \), where \( G \) is the control feedback gain matrix;

(b) the individual steady state variance constraints are satisfied, i.e.

\[
[X]_{ii} = \text{Var}(x_i(t)) \leq [X]_{ii} \leq \sigma_i^2, \quad i = 1, 2, \ldots, n \tag{5}
\]

where \( \text{Var}(x_i(t)) \) and \( \sigma_i \) denote the variance value and root mean square (RMS) constraints for the \( i \)-th state of the system, \( [X]_{ii} \) denotes the \( i \)-th diagonal element of upper bound covariance matrix \( X \), and \( [X]_{ii} \) denotes the \( i \)-th diagonal element of matrix \( X \). Here, \( X \) is defined as

\[
X = \lim_{t \to \infty} E(x(t)x^T(t)) \tag{6}
\]

(c) in the sliding mode, the \( H_\infty \) norm of the transfer matrix from \( u(t) \) to \( y(t) \) is less than a fixed scalar for system robust stability.

Goal (b) is called the UBCC problem and goal (c) can be seen as a \( H_\infty \) norm constrained problem. The presence of \( \sum_{i=1}^{P} N_i v_i(t)x(t) \) will make the UBCC problem with state feedback control be much more difficult. In order to deal with the difficulty caused by \( \sum_{i=1}^{P} N_i v_i(t)x(t) \), it is possible to utilize the invariance property of SMC to handle the problem. Consequently, the SMC will be the base to achieve goals (a)–(c) simultaneously.

### 3 SLIDING PHASE AND HITTING PHASE OF THE SYSTEM

Suppose a switching function \( S(t) \) is selected corresponding to \( x(t) \), where \( x(t) \) is the solution of equation (1) as follows

\[
S(t) = Cx(t) - \int_0^t (CA + CBG)x(\tau) \, d\tau \tag{7}
\]

where \( C \) and \( G \in \mathbb{R}^{n \times n} \) are constant matrices to be designed. \( C \) is chosen such that \( CB \) is non-singular and \( G \) is the control feedback gain matrix to be determined so that the state covariance can fit the requirement in the sliding mode. Differentiating the equation (7) with respect to time and using equation (1), \( \dot{S}(t) = CAx(t) - CBGx(t) \) is obtained. In the sliding mode, the states satisfy \( \dot{S}(t) = 0 \) \cite{23, 24}, then the equivalent control is obtained as follows

\[
u_{eq}(t) = Gx(t) - (CB)^{-1} \left[ C \sum_{i=1}^{P} N_i v_i(t)x(t) + CDw(t) \right]
\]

Substituting \( \nu_{eq}(t) \) into equation (1), the following sliding mode dynamic equation is achieved

\[
\dot{x}(t) = (A + BG)x(t) + \left( I - B(CB)^{-1}C \right) \sum_{i=1}^{P} N_i v_i(t)x(t) + Dw(t) \tag{8a}
\]

\[
y(t) = Fx(t) \tag{8b}
\]

According to the invariance property \cite{23, 24}, it is known that the system is insensitive to the perturbations in the sliding mode. Therefore, if the term \( \sum_{i=1}^{P} N_i v_i(t)x(t) \) is regarded as a perturbation to the dynamics, then the behaviour of the system in the sliding mode is insensitive to it. That is, if equation (4) holds, i.e. \( \sum_{i=1}^{P} N_i v_i(t)x(t) = B \sum_{i=1}^{P} I_i v_i(t)x(t) \), where \( I_i \in \mathbb{R}^{n \times n} \), then the second term of the right-hand side of equation (8) becomes zero. Thus, equation (8) is reduced to

\[
\dot{x}(t) = (A + BG)x(t) + Dw(t) \tag{9a}
\]
\[ y(t) = Fx(t) \] \tag{9b}

where \( \tilde{D} = (I - B(CB)^{-1}C)D \). If \( CD = 0 \) then \( \tilde{D} = D \).

Equation (9) can be rewritten as follows

\[ \dot{x}(t) = (A + BG)x(t) + Dw(t) \] \tag{10a}

\[ y(t) = Fx(t) \] \tag{10b}

Therefore, how to design a controller \( u(t) \) such that the states of the system (1) can converge to the sliding surface is shown as follows.

Define a Lyapunov function

\[ V(S(t)) = S^T(t)S(t) = S^2_1(t) + \cdots + S^2_{\gamma}(t) + \cdots + S^2_m(t) \] \tag{11}

The following theorem must be proposed first.

**Lemma 3.1** [19]

For system (1), if a Lyapunov function \( V(S(t)) \) is chosen as equation (11) and the controller \( u(t) \) is designed as equation (12)

\[ u(t) = Gx(t) - (CB)^{-1}[k||Cx(t)|| + x]sgn(S(t)) \] \tag{12}

then

\[ \frac{d}{dt} V(S(t)) = -2k||Cx(t)||||S(t)|| - 2x||S(t)|| + \text{trace} \left( \left( C \sum_{i=1}^{p} N_i x(t) \left( C \sum_{i=1}^{p} N_i x(t) \right)^T \right) \right) \]

\[ + \text{trace} \left( CDW(CD)^T \right) \] \tag{13}

In which the white noise \( w(t) \) and \( \nu(t) \) satisfy equations (2) and (3).

**Remark 2**

In equation (13), the term of \( \text{trace}(CDW(CD)^T) \) is equal to \( \text{trace}(CD(CD)^T) \) for the sake of white noise \( w(t) \) satisfying equation (2).

**Lemma 3.2** [19]

Conditioned on \( ||S(t)|| \geq ||\sum_{i=1}^{p}CN_i x(t)|| \), for the system (1) with \( CD = 0 \), set the controller \( u(t) \) to satisfy equation (12). Then the state of system will converge to the sliding surface, where \( k \geq 1 ||\sum_{i=1}^{p}N_i|| \) and \( x \) is a positive number, \( sgn(S(t)) = [sgn(S_1(t)) \ldots sgn(S_\gamma(t)) \ldots sgn(S_m(t))]^T \) in which

\[ sgn(S_j(t)) = \begin{cases} 1, & S_j(t) > 0 \\ 0, & S_j(t) = 0 \\ -1, & S_j(t) < 0 \end{cases} \]

From the above, the controller \( u(t) \) in equation (12) forces the states to hit the sliding surface. According to the matching condition (4), the invariance property exists in the sliding mode and the sliding mode dynamics is equation (10). The remaining questions are whether the matrix \( G \) exists and how to get it to achieve the goals (a)–(c)? These questions are addressed in the next section.

4 DESIGN OF CONTROL FEEDBACK GAIN MATRIX \( G \) TO ACHIEVE MULTIPLE GOALS

For the system dynamics (10) in the sliding mode, it is easy to see that state covariance \( X \), which is defined in equation (6), satisfies the following Lyapunov equation

\[ (A + BG)X + X(A + BG)^T + DD^T = 0 \] \tag{14}

Furthermore, the transfer function \( H(s) \) from noise input \( u(t) \) to output \( y(t) \) may be written as

\[ H(s) \triangleq F(sI - A - BG)^{-1}D \] \tag{15}

where \( s \) is Laplace operator. Therefore, goal (c) can be rewritten as equation (16)

\[ ||H(s)||_\infty \leq \gamma \] \tag{16}

for some prescribed positive constant \( \gamma \).

**Lemma 4.1**

Consider the system (10). Let \( G \) be given and \( \gamma > 0 \) be a fixed scalar. If there exists a positive definite matrix \( \bar{X} \) satisfying

\[ (A + BG)\bar{X} + \bar{X}(A + BG)^T + \gamma^{-2}\bar{X}\bar{R}\bar{X} + DD^T + q^{-1}(A + BG)\bar{X}(A + BG)^T + q^{-1}(q^2 - \rho^2)\bar{X} = 0 \] \tag{17}

where \( \bar{R} = F^TF \). Then the eigenvalues of \( (A + BG) \) are located within \( D(q, \rho) \) and \( ||H(s)||_\infty \leq \gamma \). Furthermore, in this case

\[ X = \bar{X}. \] \tag{18}
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Proof

Let \( \lambda \) and \( \psi \in \mathbb{C} \) (complex) be the eigenvalue and the right eigenvector of \((A + BG)\), then \((A + BG)\psi = \lambda \psi \) and \((A + BG)\psi = \lambda \psi \) in which \( \lambda = \bar{x} + j\bar{y} \) and \( \lambda = \bar{x} - j\bar{y} \). Substituting these expressions into equation (17)

\[
\psi^T [2\bar{x} + 2q^{-1}(\bar{x} + q^{-1}(q^2 - \rho^2))] \psi
\]

\[
= -\psi^T (\gamma^{-2}\bar{x}\bar{x} + DD^T) \psi
\]

\[
= [2q^{-1}(\bar{x} + q^{-1}(q^2 - \rho^2))] \psi \bar{X} \psi
\]

\[
= -\psi^T (\gamma^{-2}\bar{x}\bar{x} + DD^T) \psi
\]

\[
= [q^{-1}(\bar{x} + q^{-1}(q^2 - \rho^2))] \psi \bar{X} \psi
\]

\[
= -\psi^T (\gamma^{-2}\bar{x}\bar{x} + DD^T) \psi
\]

(19)

Since \( q > 0 \), \( \bar{X} > 0 \) and \( \gamma^{-2}\bar{x}\bar{x} + DD^T \geq 0 \), from equation (19)

\[
(x + q)^2 + y^2 - \rho^2 < 0
\]

(20)

which means that all eigenvalues of \((A + BG)\) should be located in a specified disk \( \mathcal{D}(q, \rho) \). Consider equation (17), and by the inducement of the Fact 1.2 of reference \([25]\) \( ||H(s)||_{\infty} \leq \gamma \) can be obtained. Since

\[
DD^T + q^{-1}(A + BG)\bar{X}(A + BG)^T + q^{-1}(q^2 - \rho^2)\bar{X}
\]

\( \geq 0 \), by subtracting equation (14) from equation (17) and using \( \gamma^{-2}\bar{x}\bar{x} + q^{-1}(A + BG)\bar{X}(A + BG)^T + q^{-1}(q^2 - \rho^2)\bar{X} \geq 0 \), the inequality \( X \leq \bar{X} \) will be obtained from the proofs of Theorem 4.2 of reference \([15]\) owing to the fact that \((A + BG)\) is stable.

It follows from Lemma 4.1 that the satisfaction of equation (17) leads to \((A + BG)\) is asymptotically stable; \( ||H(s)||_{\infty} \leq \gamma \), \( X \leq \bar{X} \) and \( \lambda(A + BG) \in \mathcal{D}(q, \rho) \). Therefore, in the following theorem the conditions and solutions are derived for which equation (17) is satisfied.

Theorem 4.1

Consider the system (10), given a upper bound covariance matrix \( \bar{X} = \bar{X}^T > 0 \) satisfying variance constraints (b) (i.e. \( \bar{X}_{ii} \leq \sigma_i^2 \)), then there exists a control feedback gain matrix \( G \) such that \( \bar{X} \) satisfies equation (17) if and only if

\[
(I - BB^+)\bar{X}_{\text{up}}(I - BB^+)^T = 0
\]

(21)

and

\[
(q^{-1/2}A + q^{1/2}I)\bar{X}(q^{-1/2}A + q^{1/2}I)^T - A\bar{X} - \bar{X}A^T
\]

\[
- \gamma^{-2}\bar{x}\bar{x} - DD^T - q^{-1}\bar{X}A^T
\]

\[
- q^{-1}(q^2 - \rho^2)\bar{X} \geq 0
\]

(22)

where \( \bar{X} = \Theta \Theta^T \) and \( \bar{X}_{\text{up}} = A\bar{X} + \bar{X}^T + \gamma^{-2}\bar{x}\bar{x} + DD^T + q^{-1}\bar{X}A^T + q^{-1}(q^2 - \rho^2)\bar{X} \)

Moreover, the control feedback gain solution has the following form

\[
G = B^+ \begin{bmatrix} 0 & I \\ \bar{U}_n & \bar{L}_0 \end{bmatrix} L_0^T \Theta^{-1} - (A + qI)
\]

\[
+ (I - B^+ B)Z
\]

(23)

in which \( Z \) is an arbitrary matrix, \( L \) is some orthogonal matrix (i.e. \( LL^T = I \)), \( K \) is defined as

\[
(q^{-1/2}A + q^{1/2}I)\bar{X}(q^{-1/2}A + q^{1/2}I)^T - \bar{X}_{\text{up}} = KK^T
\]

(24)

and \( U_n \) is an arbitrary orthogonal matrix with appropriate dimension; as well as \( L_n \) and \( L_0 \) are obtained from the SVD as follows

\[
(I - BB^+)K = \bar{U} \sum L_n^T
\]

(25)

\[
(I - BB^+) [q^{-1/2}A + q^{1/2}I] \Theta = \bar{U} \sum L_0^T
\]

(26)

Remark 3

The \( B^+ \) denotes the Moore–Penrose inverse of \( B \). And it satisfies with the following definitions: (i) \( BB^+B = B^+ \), (ii) \( BB^+B = B \), (iii) \( (BB^+) = BB^+ \), and (iv) \( (BB^+) = B^+B \).

Proof Necessity

Suppose there exists a \( G \) satisfying equation (17) for the given \( \bar{X} = \bar{X}^T > 0 \). Then equation (17) can be rewritten as

\[
[q^{-1/2}BG\Theta + (q^{-1/2}A + q^{1/2}I)\Theta]^T
\]

\[
[q^{-1/2}BG\Theta + (q^{-1/2}A + q^{1/2}I)\Theta]^T = KK^T
\]

(27)

From equation (27), it can be found that the right-hand side of equation (24) is positive semidefinite,
hence the condition (22) will be obtained, immediately. From equation (27)
\[ q^{-1/2}BG\Theta + (q^{-1/2}A + q^{1/2}I)\Theta = KL \] (28)
By the assumption that G satisfying equation (28) exists. Hence, equation (28) has to be solvable for G, which is guaranteed if and only if
\[ (I - BB^+) [q^{1/2}KL\Theta^{-1} - (A + qI)] = 0 \] (29)
or equivalently
\[ (I - BB^+)KL = (I - BB^+) [q^{-1/2}A + q^{1/2}I]\Theta \] (30)
Since \( \Theta \) is non-singular, using Lemma 2.1 of reference [26] and equation (30) is equivalent to
\[ (I - BB^+) [KK^T - (q^{-1/2}A + q^{1/2}I)\tilde{X}] \\
(q^{-1/2}A + q^{1/2}I) (I - BB^+)^T = 0 \] (31)
It is clear that equation (31) is equal to equation (21). From the theory of generalized inverse and equation (28), the solution G will be given by
\[ G = B^+ [q^{1/2}KL\Theta^{-1} - (A + qI)] + (I - B^+)Z \] (32)
Substituting equation (32) into equation (27) yields
\[ (A + BG)\tilde{X} + \tilde{X}(A + BG)^T + \gamma^{-2}\tilde{X}\tilde{X} + DD^T \\
+ q^{-1}(A + BG)\tilde{X}(A + BG)^T + q^{-1}(q^2 - \rho^2)\tilde{X} = 0 \] (33)
This completes the necessity of existence of G satisfying equations (21) and (22).

**Sufficiency**

Now suppose \( \tilde{X} > 0 \) satisfying equations (21) and (22). For sufficiency, it is shown that G given by equation (32) solves equation (17). From equation (32)
\[ BG = BB^+ [q^{1/2}KL\Theta^{-1} - (A + qI)] \\
= q^{1/2}KL\Theta^{-1} - (A + qI) \] (34)
Putting equation (34) into the left-hand side of equation (17) yields
\[ (A + BG)\tilde{X} + \tilde{X}(A + BG)^T + \gamma^{-2}\tilde{X}\tilde{X} + DD^T \\
+ q^{-1}(A + BG)\tilde{X}(A + BG)^T + q^{-1}(q^2 - \rho^2)\tilde{X} = 0 \]
\[ = [q^{-1/2}BG\Theta + (q^{-1/2}A + q^{1/2}I)\Theta] \\
[q^{-1/2}BG\Theta + (q^{-1/2}A + q^{1/2}I)\Theta]^T - KK^T \] (35)
Since equation (27) holds, the right-hand side of equation (35) is zero. Hence, equation (17) holds for G given by equation (32). This completes the sufficiency of the existence of G satisfying equations (21) and (22).

Moreover, from Lemma 2.1 of reference [26], the general solution L for equation (30) can be expressed as
\[ L = L_m \begin{bmatrix} I & 0 \\ 0 & U_s \end{bmatrix} L_0^T \] (36)
where \( U_s \) is an arbitrary orthogonal matrix and \( L_m \) and \( L_0 \) are expressed as in equations (25) and (26), respectively. Hence, the control feedback gain matrix G defined in equation (32) can be rewritten as equation (23). The proof is completed.

To check whether \( \|H(s)\|_\infty \leq \gamma \) holds or not, the following lemma may be helpful.

**Lemma 4.2:** [27]

Consider the system (10). There exists a positive scalar \( \gamma \) to satisfy constraint \( \|H(s)\|_\infty \leq \gamma \) if and only if \( M_s \) has no imaginary eigenvalues on the imaginary axis, where
\[ M_s = \begin{bmatrix} A + BG & -\gamma^{-1}DD^T \\ -\gamma^{-1}F^TF & -(A + BG)^T \end{bmatrix} \]

5 A NUMERICAL EXAMPLE

Consider a time-invariant bi-linear stochastic system in equation (1) with \( m = 1 \) (single input single output) and \( p = 1 \). Moreover, the system’s parameters are given as follows
\[ A = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \end{bmatrix} \]
The design goals are to find the constrained sliding mode controller such that the steady state of the system satisfies the following requirements

\[ q = 25, \quad \rho = 20 \]  
\[ \text{Var}(x_1) \leq 1.5, \quad \text{Var}(x_2) \leq 0.5 \]  
\[ ||H(s)||_\infty \leq 1 \]  

It is supposed that this system is driven by initial states \( x(0) = [x_1(0) \ x_2(0)]^T = [6 \ 6]^T \) and white noise \( w(t) \) and \( v_i(t) \), which satisfies equations (2) and (3) respectively. Then, the proposed design procedure may be obtained as follows.

**Step 1:** Choosing \( C = [1 \ 0] \) such that \( CB \neq 0 \). Clearly, the bi-linear term disappears during the sliding mode.

**Step 2:** Assigning the upper bound covariance matrix

\[ \tilde{X} = \begin{bmatrix} 1.1000 & 0.5573 \\ 0.5573 & 0.3500 \end{bmatrix} \]

which has diagonal elements satisfying the performance constraints (38).

**Step 3:** Substituting \( \tilde{X} \) into Theorem 4.1, the conditions (21), (22) are satisfied. Hence, from equation (23), the control feedback gain matrix as follows is obtained

\[ G = [ -22.9716 \ 30.8767 ] \]

**Step 4:** From equation (7), the switching function has the following form.

\[ S(t) = [1 \ 0]x(t) - \int_0^t [-22.9716 \ 31.8767]x(\tau)d\tau \]
Step 5: The controller \( u(t) \) is obtained from equation (16) as follows

\[
u(t) = \begin{bmatrix} -22.9716 & 30.8767 \end{bmatrix} x(t) - (1.2 \| Cx(t) \| + 1) \text{sgn}(S(t))
\] (42)

where \( k = 1.2 \) and \( x = 1 \) are chosen.

From the above design procedure, it can be concluded that the upper bound covariance matrix \( \tilde{X} \) will be achieved if the system is driven by controller (42) and control feedback gain matrix (40) during the sliding mode. The simulation results of the controlled states \( x_1(t) \) and \( x_2(t) \) are drawn in Fig. 1 and Fig. 2, respectively. The time responses of \( S(t) \) and \( u(t) \) are shown in Fig. 3 and Fig. 4, respectively. It is easy to check that the matrix \( M \) which defined in Lemma 4.2 has no imaginary eigenvalues, hence the \( H_{\infty} \) norm constraint (39) is satisfied. The eigenvalues of the system are also checked, which locate in \([-18.6070, -8.3646]\) which satisfy the eigenvalues placement constraint of equation (37). Moreover, from simulation the variances of \( x_1(t) \) and \( x_2(t) \) are 0.0598 and 0.0291, respectively. Therefore, the individual variance constraints (38) are also achieved.

Remark 4

From the numerical examples of referred papers [19, 20] and this numerical example, we can find that the design procedures and the structures of controls \( u(t) \) are very similar. But, using the proposed method, the multi-objective of systems requirements can be achieved in this paper.

6 CONCLUSION

This paper has proposed a method to design a controller \( u(t) \) to achieve multi-objective performance constraints for bi-linear stochastic systems. Based on the concept of sliding mode control, the linear closed-loop system can be obtained without a bi-linear term. By the utilization of sliding mode control, the designed control feedback gain matrix not only achieves the \( H_{\infty} \) norm and variance constraints for the system but also determines the sliding surface of the system. This paper presents a novel combination of sliding mode control and upper bound covariance control methods to achieve multi-objective performance constraints for bi-linear systems. Forthcoming work by the present authors is to implement this new scheme in some practical high-performance complex systems.

REFERENCES

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**APPENDIX**

**Notation**

- $\|A\|$ induced 2-norm of matrix $A$
- $A \succ B$ matrix $A-B$ is a positive semi-definite matrix
- $A > 0$ matrix $A$ is a positive definite matrix
- $\mathcal{D}(q, \rho)$ circular region with the centre at $-q+j0$ $(q > 0)$ and the radius $\rho$, $(\rho < q)$
- $\|H(s)\|_\infty$ $H_\infty$-norm of transfer matrix $H(s)$
- $I$ identity matrix
- rank($A$) rank of matrix $A$
- $R^n$ $n$-dimensional real matrix
- $R^{n \times m}$ the set of $n \times m$ dimensional real matrix
- $\text{sgn}(\cdot)$ sign function of $(\cdot)$
- $\|S(t)\|$ 2-norm of vector $S(t)$
- $\|S(t)\|_1$ 1-norm of vector $S(t)$
- $\text{trace}(A)$ trace of matrix $A$
- $\lambda(A)$ eigenvalues of matrix $A$
- $\psi$ right eigenvector of matrix $(\cdot)$
- $\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathcal{P}$ a complete probability space with a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$
- $(\cdot)^T$ transposition of $(\cdot)$
- $(\cdot)^*$ Moore–Penrose inverse of $(\cdot)$
- $(\cdot)^\dagger$ conjugate transpose of $(\cdot)$