Gain-scheduled control of PVTOL aircraft dynamics with parameter-dependent disturbance

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Abstract

This paper presents a gain-scheduled control approach for the vertical takeoff and landing aircraft. The non-linear aircraft dynamics are formulated as a linear parameter varying (LPV) system with external parameter-dependent disturbance, which arisen from the equilibrating between gravity force and nozzles thrust. The disturbance is dependent on the system varying parameter, roll angle, and a constant parameter denoting the normalized gravity force. The controllers are designed in terms of mixed optimization of $\mathcal{H}_\infty$ performance for disturbance attenuation and relative stability for tracking position command in pitch-yaw plane. The characteristics of the parameter-dependent disturbance are described by an equality condition with a defined annihilation matrix. By exploring the parameter-dependence condition on disturbance into the controller design algorithms based on linear matrix inequalities (LMIs), it is showed that a better performance can be achieved than simply considering it as an external disturbance. The design results are demonstrated by time response simulations.

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The vertical/short takeoff and landing (V/STOL) aircraft has the capability of high mobility and maneuverability, which typically is equipped with exhaust nozzles in each side of the fuselage to provide the gross thrust for the aircraft. The nozzles are capable of rotating together from the aft position forward to near 100°. This change in the direction of thrust allows the aircraft to operate in two modes and the transition between them. In the mode of wing-borne forward flight, the nozzles are in the aft position and the produced aerodynamic forces by the surfaces of wing and fuselage sustain the weight of aircraft. In the mode of jet-borne hovering, the nozzles are directed vertically to the ground. Moreover, the transit process between these two modes enables the maneuvering in velocity and direction of the aircraft.

For the hovering operation considered in this paper, the upward thrust produced by nozzles is manipulated by throttle to perform the aircraft motion in vertical direction. In order to perform motion in lateral direction, the aircraft is equipped with reaction control valves in the nose, tails, and wingtips. If the air from the reaction control valves for producing moment in attitude control induces an unexpected force component in the lateral direction, the system dynamics may exhibit non-minimum phase characteristics which depend on the direction of induced force. In the case that the moment changing the aircraft attitude induces a force that moves the aircraft into an opposite lateral direction, then this inevitable time delay will cause the non-minimum phase effect.

Several non-linear controller design approaches have been conducted for this non-linear non-minimum phase planar V/STOL aircraft dynamics. One possibility is based on the well-known input–output feedback linearization approach which implies a non-linear version of pole-zero cancellation [1,2]. For a non-minimum phase system, this feedback linearization will result in internally unstable dynamics; even the linearized system is stable in the sense of input–output stability. To avoid this internal instability, feedback linearization was performed based on approximated minimum phase dynamics, in which the influence of rolling moment on the lateral force was neglected [3]. It was shown that the desired properties such as bounded tracking and asymptotic stability for the true system are maintained if the neglected parasitic coupling effect is small.

Recently in [4], a non-linear robust state-feedback control law was designed for the planar V/STOL aircraft by using an optimal control approach. In [5], the non-minimum phase planar V/STOL dynamics was considered as combination of a linear part resulted from input–output linearization and a non-linear part denoting the dynamics that does not depend explicitly on the inputs. Then, a composite non-linear state feedback control law based on Lyapunov technique with minimum-norm strategy was developed to stabilize the overall closed-loop system. The authors in [6] proposed a non-linear controller to achieve globally asymptotical stabilization of the planar V/STOL aircraft and implemented the algorithm in a real-time application. In the newly published work [7], a non-linear prediction-based control approach was proposed, in which a partial feedback linearization was performed and the optimal trajectories were proposed for the linearized variables. It should be noted that in all the approaches of [4–7], the non-minimum phase parasitic effect was explicitly utilized in the feedback control law. However, the non-minimum phase effect is desirable to be considered as an uncertain bounded factor, as proposed in this work, instead of a measurable parameter.
Another approach for the V/STOL aircraft control is to design a family of controllers beforehand according to the operation envelope of aircraft system. Then, in the real-time application, a mechanism for scheduling among this family of controllers is activated to realize the instantaneous controller based on the aircraft operation. In [8], based on the generic V/STOL aircraft model (GVAM), a gain-scheduled design using the $H^\infty$ optimal control technique is investigated. In [9], a composite control law approach is proposed for the planar V/STOL aircraft dynamics. A state-feedback control law is designed by using linear matrix inequality (LMI) method [10] to achieve robustness against the parasitic uncertainty without considering the gravitational disturbance on aircraft dynamics. Then, another non-linear control law is constructed to handle this gravitational disturbance effect.

In this paper, the non-linear planar V/STOL dynamics is formulated as a linear parameter varying (LPV) system [11–13] with external parameter-dependent disturbance, which arisen from the equilibrating between the gravity force and nozzles thrust. The system varying parameter is the roll angle, which is also one of the states and considered as the scheduling variable that specifies the operating point of the aircraft system. The disturbance is dependent on the roll angle as well as a constant parameter denoting the normalized gravity force. In the framework of LMI approach, the design of this gain-scheduled control is conducted in terms of the mixed optimization of $H^\infty$ performance for the disturbance attenuation and relative stability for position tracking in the pitch-yaw plane. Particularly as shown in this paper, the parameter-dependent disturbance can be characterized by some equality conditions such that in the LMI controller design algorithms, some proper annihilation conditions can be derived to improve the resulting design performance.

The remainder of this paper is organized as follows. Section 2 presents the modeling of planar V/STOL aircraft dynamics and the reformulation as an LPV system with parameter-dependent disturbance. Section 3 is for design of the proposed gain-scheduled control based on LMI method. Then, Section 4 is the simulations and discussions. Section 5 is the conclusion.

2. Planar V/STOL aircraft dynamics

Considering the typical V/STOL aircraft, the thrust vector provided by the throttle and nozzle enables two-degrees-of-freedom control in the pitch-yaw plane. In order to allow lateral maneuverability in the jet-borne operation, the aircraft also has a reaction control system (RCS) to provide moment around the aircraft center of mass. In the case that the bleed air from the reaction control valves produces force which is not perpendicular to the pitch axis, there will be a coupling effect between the angle rolling moment and lateral moving force.

By restricting the aircraft to the jet-borne operation, i.e., thrust directed to the bottom of the aircraft, we have simplified dynamics which describes the motion of the aircraft in the vertical—lateral directions, i.e., a planar V/STOL (PVTOL) aircraft as shown in Fig. 1. The aircraft states are the position of center of mass, $(X, Y)$, the roll angle $\theta$, and the corresponding velocities, $(\dot{X}, \dot{Y}, \dot{\theta})$. The control input is the thrust directed to the bottom of aircraft $U_1$ and the moment around the aircraft center of mass $U_2$. Let the quantity of lateral force induced by rolling moment be denoted by $e_0$, then we have the aircraft
dynamics written as
\[
\begin{align*}
-m\ddot{X} &= -\sin \theta U_1 + \varepsilon_0 \cos \theta U_2, \\
-m\ddot{Y} &= \cos \theta U_1 + \varepsilon_0 \sin \theta U_2 - mg, \\
J\ddot{\theta} &= U_2,
\end{align*}
\]
where \(mg\) is the gravity force imposed in the aircraft center of mass and \(J\) is the mass moment of inertia around the axis through the aircraft center of mass and along the fuselage.

To simplify the notation of the PVTOL aircraft dynamics (1), the first and second equations in Eq. (1) are divided by \(mg\) and the third one by \(J\). Let \(x = -X/g\), \(y = -Y/g\), \(u_1 = U_1/\varepsilon_0\), \(u_2 = U_2/\varepsilon_0\), \(\varepsilon = \varepsilon_0 J/mg\), then we have the normalized PVTOL aircraft dynamics
\[
\begin{align*}
\dot{x} &= - \sin \theta u_1 + \varepsilon \cos \theta u_2, \\
\dot{y} &= \cos \theta u_1 + \varepsilon \sin \theta u_2 - 1, \\
\dot{\theta} &= u_2.
\end{align*}
\]

The term \(\varepsilon \dot{y}\) denotes the normalized gravity acceleration. The coefficient \(\varepsilon\) denotes the parasitic coupling effect between the control moment and lateral force, which results in the non-minimum phase characteristic.

The non-linear PVTOL model (2) can be rewritten as an uncertain LPV system with dependence on measurable varying roll angle \(\theta\) and uncertain coupling parameter \(\varepsilon\)
\[
\dot{x}_v = A_v x_v + (B_1(\theta) + \varepsilon B_2(\theta))u + D,
\]
where \(x_v = (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})^T\), \(u = (u_1, u_2)^T\). In the system matrix \(A_v\), the elements \(A_v(1,2)\), \(A_v(3,4)\) and \(A_v(5,6)\) are equal to 1, and others are zeros. The parameter-dependent input matrices are
\[
B_1(\theta) = \begin{pmatrix} 0 & -\sin \theta & 0 & \cos \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T, \quad B_2(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta & 0 & 0 \end{pmatrix}^T
\]
and the disturbance matrix \(D = (0,0,0\ldots,0,0)^T\) represents the gravity acceleration.
It is noted that the state dynamics \((x, \dot{x}), (y, \dot{y}), \) and \((\theta, \dot{\theta})\) are decoupled to each other by the structure of system matrix \(A_v\). Also, the nominal input matrix \(B_1(\theta)\), which is not related to the parasitic parameter \(\varepsilon\), corresponding to lateral position dynamics \((x, \dot{x})\) is a 2-by-2 zero matrix at the equilibrium value \(\theta = 0\). Therefore, the LPV representation of the PVTOL aircraft (3) is considered as uncontrollable at the equilibrium point \(\theta = 0\) for the lateral position dynamics even for the case of nominal aircraft system with \(\varepsilon = 0\).

The following procedures are performed to remedy this difficulty during the LPV modeling (3) for original non-linear PVTOL dynamics (2):

- Let the variable of thrust \(u_1\) be centered around equilibrium point “1”, i.e., \(u_1 := 1 + \hat{u}_1\). The revised LPV system of Eq. (3) is

\[
\dot{x}_v = A_v x_v + (B_1(\theta) + \varepsilon B_2(\theta))\hat{u} + \hat{D}(\theta),
\]

where \(\hat{u} = (\hat{u}_1, u_2)^T\) and \(\hat{D}(\theta) = (0, -\sin \theta, 0, -1 + \cos \theta, 0, 0)^T\).

- Let the vector \(\hat{D}(\theta)\) decomposed as \(\hat{D}(\theta) = A_{v0}x_v + \hat{D}(\theta)\). In the matrix \(A_{v0}\), the only non-zero element is \(A_{v0}(2, 5) = -1\). In the vector \(\hat{D}(\theta) = (0, \theta - \sin \theta, 0, -1 + \cos \theta, 0, 0)^T\), the elements “\(\theta - \sin \theta\)” and “\(-1 + \cos \theta\)” represent the first-order Taylor series approximation errors of the trigonometric functions “\(-\sin \theta\)” and “\(\cos \theta\)”, respectively. Then, the LPV PVTOL aircraft dynamics (5) is rewritten as

\[
\dot{x}_v = A_0 x_v + (B_1(\theta) + \varepsilon B_2(\theta))\hat{u} + \hat{D}(\theta),
\]

where \(A_0 := A_v + A_{v0}\), and \(\hat{D}(\theta)\) is considered as a gravitational disturbance.

It is noted that the introduced non-zero element of \(A_{v0}\) enables the lateral position variables \((x, \dot{x})\) to be controlled through the \((\theta, \dot{\theta})\) dynamics.

The gravitational disturbance \(\hat{D}(\theta)\) in Eq. (6) can be decomposed as linear combinations of parameters and denoted as, \(\hat{D}(\theta) = A_{\xi} \zeta(\theta)\), where

\[
A_{\xi} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \zeta(\theta) = \begin{pmatrix}
\theta \\
\sin \theta \\
\cos \theta \\
1
\end{pmatrix}
\]

and \(\zeta(\theta)\) is considered as parameter-dependent disturbance. Therefore, the resulting PVTOL aircraft dynamics become

\[
\dot{x}_v = A_0 x_v + (B_1(\theta) + \varepsilon B_2(\theta))\hat{u} + A_{\xi} \zeta(\theta).
\]

It is noted that the parameter-dependent input matrix \(B_1(\theta)\) as shown in Eq. (4) can be denoted as an affine function of the varying parameters, \(p_\xi := \sin \theta\) and \(p_\zeta := \cos \theta\),

\[
B_1(\theta) := B_0 + p_\xi B_s + p_\zeta B_c,
\]
where

\[
B_0 = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \quad B_s = \begin{pmatrix}
0 & 0 \\
-1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad B_c = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\] (10)

Moreover, by the algebraic relationship existing between the varying parameters, \( p_s \) and \( p_c \), we can find suitable parameter-dependent matrix \( N_z(y) \) such that the equality condition on the parameter-dependent disturbance, \( z(y) \)

\[
N_z(y)z(y) = 0 \quad (11)
\]
is satisfied. One choice of this annihilation matrix \( N_z(\theta) \) is

\[
N_z(\theta) = \begin{pmatrix}
0 & \sin \theta & \cos \theta & -1 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & 1 & 0 & -\sin \theta \\
0 & 0 & 1 & -\cos \theta
\end{pmatrix} \equiv N_0 + p_s N_s + p_c N_c \quad (12)
\]
with

\[
N_0 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad N_s = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad N_c = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\] (13)

Then, the complete PVTOL aircraft dynamics includes the state-space equation in Eq. (8) and the algebraic equation in Eq. (11), and can be jointly represented as a descriptor system.

3. Gain-scheduled controller design via LMI optimization

The gain-scheduled control for the PVTOL aircraft is based on the dynamics in Eq. (8) and the equality condition in Eq. (11). The design objective is to maintain certain degree of relative stability for all admissible values of the measurable varying parameter \( \theta \) and its variation rate \( \dot{\theta} \), the expected amount of the uncertain parasitic coupling \( \epsilon \), while subject to physical limitations on the magnitude of the control efforts, \(|\tilde{u}_1|\) and \(|u_2|\). Moreover, the effect of the parameter-dependent disturbance \( \zeta(\theta) \) on the system response needs to be minimized or maintained in a certain level.

To designate the effect of the disturbance on the system output, the regulated output is chosen as

\[
z = \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} x_v := C_z x_v. \quad (14)
\]
Then, the design dynamics includes the state-space equation in Eq. (8) and the regulated output in Eq. (14)
\[
\dot{x}_v = A_0 x_v + (B_1(\theta) + \varepsilon B_2(\theta))\hat{u} + A_{\zeta} \zeta(\theta), \quad z = C_2 x_v, \tag{15}
\]
the equality condition on the disturbance in Eq. (11), \( N_2(\theta) \zeta(\theta) = 0 \), and the inequality magnitude constraints on the control efforts \((\hat{u}_1, u_2)\) and varying parameters\((\theta, \dot{\theta})\),
\[
|\hat{u}_1| \leq u_{1,\text{max}}, \quad |u_2| \leq u_{2,\text{max}}, \quad |	heta| \leq \theta_{\text{max}}, \quad |\dot{\theta}| \leq \dot{\theta}_{\text{max}}. \tag{16}
\]
Also, for the roll angle \(\theta\) with bounded magnitude \(\theta_{\text{max}}\), let the ranges of the varying parameters \((p_s, p_c)\) denoted as
\[
p_s \in [-\sin \theta_{\text{max}}, \sin \theta_{\text{max}}] = [p_s, \bar{p}_s], \quad p_c \in [\cos \theta_{\text{max}}, 1] = [p_c, \bar{p}_c]. \tag{17}
\]

### 3.1. \(\mathcal{H}_\infty\) Optimization for relative stability and disturbance rejection

Consider the controller synthesis structure as shown in Fig. 2, where \(\mathcal{P}(\theta, \varepsilon)\) denotes the PVTOL dynamics in Eq. (15) with control input \(\hat{u}\), disturbance \(\zeta(\theta)\), and regulated output \(z\). This paper is on the gain-scheduled design with state-feedback control law \(\hat{u} = K(\theta)x_v\), such that the controlled system of the PVTOL dynamics in Eq. (15) subject to the physical constraints addressed in Eqs. (16) and (17) can achieve certain levels of maximum relative stability and disturbance rejection performance. Specifically, the annihilation condition (11) on the parameter-dependent disturbance \(\zeta(\theta)\) will be investigated to explore the potential performance improvement over the design by simply considering \(\zeta(\theta)\) as an external disturbance. In this paper, the positive parameter-dependent quadratic Lyapunov function \(V(x_v, \theta) = x_v^T P(\theta)x_v, P(\theta) > 0\) is assumed.

The effect of the disturbance \(\zeta(\theta)\) on the regulated output \(z\) of the closed-loop controlled system is designated by the \(\mathcal{H}_\infty\) performance level constraint
\[
\|T_{\zeta}\|_\infty = \sup_{\zeta(\theta) \neq 0} \frac{\|z(t)\|_2}{\|\zeta(t)\|_2} = \sup_{\omega} \sigma(T_{\zeta}(j\omega)) < \gamma. \tag{18}
\]

According to the Bounded-Real Lemma [10] and without considering the parameter-dependent characteristics of the disturbance \(\zeta(\theta)\), the performance constraint (18) for the
design dynamics (15) is equivalent to the following matrix inequality condition:

\[ \Pi(\theta, \varepsilon, \mu, \gamma) := \begin{pmatrix} A_{cl}(\theta)^T P(\theta) + P(\theta) A_{cl}(\theta) + \dot{P}(\theta) - 2\mu P(\theta) & (*) & (*) \\ (*) & -\gamma I & (*) \\ (*) & 0 & -\gamma I \end{pmatrix} < 0, \]  

where \( A_{cl}(\theta) := A_0 + (B_1(\theta) + \varepsilon B_2(\theta))K(\theta) \) and (*) denotes for matrix symmetric component. Pre-multiplying and post-multiplying by the block-diagonal matrix \( \text{diag}(P(\theta)^{-1}, I, I) \), then denoting \( P(\theta)^{-1} = Q(\theta), K(\theta)Q(\theta) = L(\theta) \), and from the identity

\[ P(\theta)^{-1}\dot{P}(\theta)P(\theta)^{-1} = -\dot{P}(\theta)^{-1} = -\dot{Q}(\theta), \]  

the condition of the matrix inequality (19) becomes

\[ \Sigma(\theta, \varepsilon, \mu, \gamma) := \begin{pmatrix} \Xi(\theta) + \varepsilon B_2(\theta)L(\theta) + \varepsilon L(\theta)^T B_2(\theta)^T & (*) & (*) \\ -\dot{Q}(\theta) - 2\mu Q(\theta) & A_{cl}(\theta)^T & -\gamma I \\ (*) & C_{cl}(\dot{\gamma}) & -\gamma I \end{pmatrix} < 0, \]  

where \( \Xi(\theta) = Q(\theta)A_0^T + A_0Q(\theta) + B_1(\theta)L(\theta) + L(\theta)^T B_1(\theta)^T \).

The input matrix \( B_1(\theta) \) is denoted as affinely parameter-dependent matrix as shown in Eqs. (9) and (10). The input matrix \( B_2(\theta) \) which multiplied by the uncertain coefficient \( \varepsilon \) is denoted by

\[ B_2(\theta) = B_2(\gamma_0 + \gamma(\theta))B_{2r} := B_{20} + B_{2l}\gamma(\theta)B_{2r}, \]  

where

\[ B_{2l} = \begin{pmatrix} 0 & p_c - 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{p}_s & 0 & 0 \end{pmatrix}^T, \quad B_{2r} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \]

\[ \gamma_0 = \begin{pmatrix} 1 \\ p_c - 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma(\theta) = \begin{pmatrix} p_c - 1 \\ \bar{p}_s - 1 \\ 0 \\ 0 \\ \bar{p}_s \end{pmatrix}. \]  

In the parameter-dependent matrix \( \gamma(\theta) \), the elements \( \gamma(\theta)(1,1) \in [0,1], \gamma(\theta)(2,2) \in [-1,1] \) and \( \gamma(\theta) \) satisfies the norm-bounded condition

\[ \gamma(\theta)^T(\theta) \leq I, \quad \forall \theta \leq \theta_{\max}, \]

which yields the following inequality:

\[ \begin{pmatrix} I & \gamma(\theta) \\ \gamma^T(\theta) & I \end{pmatrix} \geq 0, \]

by applying Schur complements [10]. For any non-zero number \( \zeta \), we have

\[ \begin{pmatrix} \zeta B_{2l} & -\zeta^{-1}L(\theta)^T B_{2r}^T \end{pmatrix} \begin{pmatrix} I & \gamma(\theta) \\ \gamma^T(\theta) & I \end{pmatrix} \begin{pmatrix} \zeta B_{2l} & -\zeta^{-1}L(\theta)^T B_{2r}^T \end{pmatrix} \geq 0. \]
and
\[ B_{2l} l(\theta) B_{2l} L(\theta) + (B_{2l} l(\theta) B_{2l} L(\theta))^T \leq \zeta^2 B_{2l} B_{2l}^T + \zeta^{-2} L(\theta)^T B_{2l}^T B_{2l} L(\theta). \]  
(27)

By applying Schur complements again to (1, 1) element of Eq. (21), the sufficient condition of Eq. (21) can be obtained as
\[
\Sigma(\theta, \varepsilon, \zeta, \mu, \gamma) := \begin{pmatrix}
\Xi(\theta) + \varepsilon B_{2l} L(\theta) + \varepsilon L(\theta)^T B_{2l}^T + \varepsilon \zeta^2 B_{2l} B_{2l}^T - \zeta^{-1} L(\theta)^T B_{2l}^T L(\theta) \\
B_{2l} L(\theta) - \varepsilon^{-1} \zeta^2 I \\
A_{\zeta}^T \\
C_{\zeta}^T Q(\theta) 
\end{pmatrix} < 0.
\]  
(28)

Incorporated with the annihilation condition on \( \zeta(\theta) \) as shown in Eq. (11), we can find an augmented condition to the parameterized matrix inequality in Eq. (19) as follows:
\[
\Pi(\theta, \varepsilon, \zeta, \mu, \gamma) + \begin{pmatrix}
N_{\zeta}(\theta)^T M_1^T P(\theta) M_2 N_{\zeta}(\theta) + (\ast) (\ast) (\ast)
0
0
0
\end{pmatrix} < 0,
\]  
(29)

where \( M_1 \in \mathbb{R}^{6 \times 4}, M_2 \in \mathbb{R}^{4 \times 4} \) are introduced extra matrix variables associated with the annihilation matrix \( N_{\zeta}(\theta) \). By pre-multiplied by \((x, z(\theta)^T x^T)\) and post-multiplied by its transpose, respectively, and from the conditions in Eq. (11), the two matrix inequalities (19) and (29) are shown to be equivalent. However, the achieved performance by using the matrix inequality (11) can be effectively improved due to the extra degrees of freedom introduced by the matrix variables \( M_1 \) and \( M_2 \). Then proceeding with pre-multiplying and post-multiplying by the block-diagonal matrix \( \text{diag}(P(\theta)^{-1}, I, I) \) to Eq. (29), the matrix inequality (28) can be refined as
\[
\Sigma(\theta, \varepsilon, \zeta, \mu, \gamma) + \begin{pmatrix}
0 (\ast) (\ast) (\ast)
N_{\zeta}(\theta)^T M_1^T M_2 N_{\zeta}(\theta) + (\ast) (\ast)
0
0
0
\end{pmatrix} < 0.
\]  
(30)

For the affinely parameter-dependent matrix \( B_1(\theta) \) in Eq. (9), the matrix variables \( Q(\theta) \) and \( L(\theta) \) can be assumed to have the same affine dependence on the varying parameters \( p_s \), \( p_c \), and then proceeded with the approach presented in [14]. However, for the parameter-independent system matrix \( A_0 \) and parameter-dependent control law \( K(\theta) = L(\theta)Q(\theta)^{-1} \) to be concerned, a single parameter-independent matrix variable for \( L(\theta) \) is chosen to reduce the numerical complexity. We have
\[
Q(\theta) = Q_0 + p_s Q_s + p_c Q_c > 0, \quad L(\theta) = L.
\]  
(31)

Then, the matrix inequality on \( \Sigma(\theta, \varepsilon, \zeta, \mu, \gamma) \) as shown in Eq. (28) reads
\[
\begin{pmatrix}
\Xi(\theta) + \varepsilon B_{2l} L + \varepsilon L^T B_{2l}^T + \varepsilon \zeta^2 B_{2l} B_{2l}^T - \zeta^{-1} \varepsilon^2 I (\ast) (\ast) (\ast)
B_{2l} L - \varepsilon^{-1} \varepsilon \zeta^2 I (\ast) (\ast)
A_{\zeta}^T \\
C_{\zeta}^T Q(\theta)
\end{pmatrix} < 0.
\]  
(32)
Since the matrix inequality (32) is affinely dependent on the parameters \( \{p_s, p_c\} \) when \( \dot{\theta} \) is fixed and also affinely dependent on \( \dot{\theta} \) when \( \{p_s, p_c\} \) is fixed, (32) is convex along each direction of \( \{p_s, p_c\} \) and \( \dot{\theta} \), respectively. The feasibility of Eq. (32) as well as Eqs. (28) and (30) can be established by evaluation only in the extreme values of the scalar varying parameters \( \{p_s, p_c, \dot{\theta}\} \) and recast as the type of generalized eigenvalue minimization problem (GEVP) [15] as follows:

\[ \text{Maximize the achievable relative stability } \alpha = -\mu \text{ under given disturbance attenuation level } \gamma = \gamma_0 \]

\[ \text{minimize } \mu : (31), (28) \text{ or } (30), \quad (33) \]

\[ \text{Minimize the disturbance attenuation level } \gamma \text{ under given degree of relative stability } \alpha = -\mu_0 \]

\[ \text{minimize } \gamma : (31), (28) \text{ or } (30) \quad (34) \]

for \( \{p_s, p_c, \dot{\theta}\} \in \{(p_s, p_s), (p_c, p_c), (-\theta_{\text{max}}, \theta_{\text{max}})\} \) with variables \( \{Q_0, Q_s, Q_c, L, \zeta\} \) in Eq. (28) and with variables \( \{Q_0, Q_s, Q_c, L, \zeta, M_1, M_2\} \) in Eq. (30).

### 3.2. System constraints on varying parameters and control inputs

The objective of maximized relative stability or disturbance attenuation in terms of \( H^\infty \) performance will tend to have a controller with high gain. The required control effort may beyond the magnitude limits of the system actuators. Moreover, the behavior of the controlled system with high gain controller will be sensitive to the values of system varying parameters. Therefore, in designing controller with maximum relative stability or disturbance attenuation, we need to consider the issues related to the system varying parameters and the physical constraints on actuator as well. The magnitude constraints on control efforts \( \tilde{u}_i \), roll angle \( \theta \), and variation rate of roll angle \( \dot{\theta} \) as shown in Eq. (16), can be addressed based on the invariant ellipsoid interpretation [10] of the parameter-dependent quadratic Lyapunov matrix \( P(\theta) > 0 \) such that \( x_{v0}^T P(\theta) x_{v0} \leq 1 \) for some initial condition \( x_{v0} \), which in turn can be written as the LMI constraint in the affinely parameter-dependent matrix variable \( Q(\theta) = P(\theta)^{-1} \)

\[
\begin{pmatrix}
1 & x_{v0}^T \\
X_{v0} & Q(\theta)
\end{pmatrix} \succeq 0.
\]

(35)

If \( Q(\theta) \) is indeed also a stabilizing solution to the GEVP described in Eq. (33) or (34), then \( x_{v}^T P(\theta) x_{v} < x_{v0}^T P(\theta) x_{v0} \leq 1 \). The magnitude constraints \( |\tilde{u}_i| \leq \tilde{u}_{i,\text{max}} \) can be satisfied if

\[
\frac{1}{\tilde{u}_{i,\text{max}}} x_{v}^T Q(\theta)^{-1} L_i^T L_i Q(\theta)^{-1} x_{v} \leq x_{v}^T Q(\theta)^{-1} x_{v},
\]

(36)
where $L_i \in \mathbb{R}^{1 \times 6}$ is the $i$th row of the matrix $L$. The conditions in Eq. (36) can be written as the following LMIs for given $u_{i,\text{max}}$:

$$
\begin{pmatrix}
Q(\theta) & L^T s_i^T \\
L s_i & u_{i,\text{max}}^2
\end{pmatrix} \succeq 0,
$$

(37)

where the row vector $s_i \in \mathbb{R}^{1 \times 2}$ with the $i$th element equals to 1 and others are zeros.

Similarly, the magnitude constraints on the roll angle, $|\theta| \leq \theta_{\text{max}}$, and variation rate of roll angle, $|\dot{\theta}| \leq \dot{\theta}_{\text{max}}$, can be met if

$$
\frac{1}{\theta_{\text{max}}^2} x_v^T r_5^T r_5 x_v \leq x_v^T Q(\theta)^{-1} x_v, \quad \frac{1}{\dot{\theta}_{\text{max}}^2} x_v^T r_6^T r_6 x_v \leq x_v^T Q(\theta)^{-1} x_v,
$$

(38)

where the row vector $r_j \in \mathbb{R}^{1 \times 6}$ with the $j$th element equals to 1 and others are zeros. The conditions in Eq. (38) can be written as the following LMIs for given $\theta_{\text{max}}$ and $\dot{\theta}_{\text{max}}$:

$$
\begin{pmatrix}
Q(\theta) & Q(\theta) r_5^T \\
r_5 Q(\theta) & \theta_{\text{max}}^2
\end{pmatrix} \succeq 0, \quad \begin{pmatrix}
Q(\theta) & Q(\theta) r_6^T \\
r_6 Q(\theta) & \dot{\theta}_{\text{max}}^2
\end{pmatrix} \succeq 0.
$$

(39)

4. Simulations and discussions

4.1. Controller numerical construction

The gain-scheduled controller designs for input and output constrained PVTOL aircraft dynamics with parameter-dependent disturbance via LMI optimization are numerically constructed in the following. When the disturbance $\zeta(\theta)$ is simply considered as an external input by ignoring its dependence on the system variable $\theta$, the matrix inequality for $H^\infty$ performance (28) is used. On the other hand, when effect of this parameter dependence is concerned, the inequality (30) with extra matrix variables $M_1$ and $M_2$ to address the annihilation condition (11) on disturbance $\zeta(\theta)$ is utilized.

The LMI conditions (28) or (30) are parameterized by the scheduling variables $\{p_c, p_c\}$ and performed to either maximize the achievable relative stability $\alpha = -\mu$ for given certain disturbance attenuation level $\gamma = \gamma_0$ by using the algorithm (33) or minimize the disturbance attenuation level $\gamma$ for given degree of relative stability $\alpha = -\mu_0$ by alternatively using Eq. (34). Also, the LMIs related to the various magnitude constraints on control efforts $u_i$ described in Eqs. (35) and (37), and on the varying parameters roll angle $\theta$ and its variation rate $\dot{\theta}$ in Eq. (39) need to be addressed concurrently for the parameter vertices, $\{p_s, p_c, \dot{\theta}\} \in \{\mathbb{R}, \mathbb{R}_s, \mathbb{R}_c\}, \{-\dot{\theta}_{\text{max}}, \dot{\theta}_{\text{max}}\}$. Since the desired objective is the tracking of the normalized position command in the horizontal and vertical direction, the initial conditions used in the parameter-dependent invariant ellipsoid (35) are specified as $x_{v0} = (\pm 1.0, \pm 1.0, 0, 0)^T$.

In first, the nominal performance with $\varepsilon = 0$ in the LMI, Eqs. (28) and (30) is investigated. The resulting design performances in terms of the achieved maximum relative stability $\alpha^* = -\mu^*$ under given disturbance level $\gamma_0 = 10$ by using the design algorithm (33) is shown in Fig. 3, and alternatively the achieved minimum disturbance level $\gamma^*$ under given relative stability $\alpha_0 = 0.1$ by using Eq. (34) is shown in Fig. 4. In these designs,
Fig. 3. Maximum relative stability $\alpha = -\mu$ under parameter-dependent disturbance $\zeta(\theta)$.

Fig. 4. Minimum disturbance attenuation level $\gamma$ under parameter-dependent disturbance $\zeta(\theta)$.
magnitude constraint on the deviated thrust is specified as \( u_{1,\text{max}} = 0.6 \), constraints on the moment are used as design parameters and chosen as \( u_{2,\text{max}} = \{0.2, 0.4, 0.6, 0.8, 1.0, 1.2\} \), the roll angle and its variation rate are designated as \( \theta_{\text{max}} = \theta_{\text{max}} = \pi u_{2,\text{max}}/k \), for \( k \in \{3, 4, 5, 6\} \). In Figs. 3 and 4, the illustrated curves for different values of \( k \) are denoted by \( \{\Delta: k = 3, \Diamond: k = 4, \times: k = 5, \nabla: k = 6\} \).

The chosen magnitude constraints on \( \theta_{\text{max}} \) and \( \dot{\theta}_{\text{max}} \) are proportional to \( u_{2,\text{max}} \) since the control moment \( u_2 \) needs a certain range of \( \theta \) and \( \dot{\theta} \) for manipulating motion in the horizontal plane. However, the quantities of \( \theta_{\text{max}} \) and \( \dot{\theta}_{\text{max}} \) correspond to the extreme values of the scalar varying parameters \( \{p_s, p_c, \theta\} \) that need to be addressed in the matrix inequality constraints \( \{(28) \text{ or } (30)\} \) and \( \{(35),(37),(39)\} \). As seen from Fig. 3, a looser attitude maneuverability constraint, typically \( k = 3 \), tends to admit a faster position tracking performance when a small magnitude of control moment is available. On the other hand, a tighter attitude maneuverability, said \( k = 6 \), tends to maintain faster position tracking when a large magnitude of control moment is available. Similar interpretation can be found from the achievable minimum disturbance attenuation levels as shown in Fig. 4. Specifically as shown in Fig. 3, the obtained maximum relative stabilities are negative or near zero for the designs with \( u_{2,\text{max}} = \{0.2\} \) under \( k \in \{4, 5, 6\} \) and \( u_{2,\text{max}} = \{1.2\} \) under \( k \in \{3\} \), in which the feasible set for an acceptable relative stability is empty and the associated designs can not be used.

In Figs. 3 and 4 for the nominal designs with \( \varepsilon = 0 \), the dotted lines in the left subplots are depicted for the designs by using Eq. (28), and the achieved performance with the parameter dependence of the disturbance explicitly addressed by using Eq. (30) are denoted by solid lines in the right subplots. As seen from the comparisons of the resulting curves, the proposed approach by including extra design variables for the property of the parameter dependence on disturbance can achieve higher maximum relative stabilities as shown in Fig. 3 and lower disturbance attenuation levels as shown in Fig. 4.

In considering the robustness issue related to the parasitic uncertainty represented by the moment–force coupling factor \( \varepsilon \), the designs in Figs. 3 and 4 with \( \theta_{\text{max}} = \dot{\theta}_{\text{max}} = \pi u_{2,\text{max}}/5 \) are selected and constructed for controller with uncertainty \( \varepsilon = 1 \) by using the matrix inequalities (28) and (30), respectively. The design results are shown in Fig. 5. The curves denoted by “□” are for the robust designs with \( \varepsilon = 1 \) and denoted by “×” are the nominal designs with \( \varepsilon = 0 \) taken from Figs. 3 and 4 for comparison. The solid lines are depicted for designs by Eq. (30) with annihilation matrix \( N_z(\theta) \) addressed and the dotted lines are for the designs by Eq. (28) without considering the annihilation condition. In Fig. 5, as seen from the maximum relative stabilities in the left subplots and the minimum disturbance levels in right subplots, the enhanced designs by using Eq. (30) can achieve better design performances than using Eq. (28) for both the case of nominal design with \( \varepsilon = 0 \) and robust design with \( \varepsilon = 1 \).

### 4.2. Time response simulations

Simulations were conducted for the closed-loop controlled system as shown in Fig. 6, where the non-linear PVTOL dynamics is from Ref. (3) with \( B(\theta, \varepsilon) := B_1(\theta) + \varepsilon B_2(\theta) \). The command generator is a low-pass filter with bandwidth at 100 rad/s for both the lateral command \( x_d \) and vertical command \( y_d \). Therefore, the actual tracking command \( (x_{\text{tr}}, y_{\text{tr}}) \) issued to the controlled system is generated from the exogenous command signal \( (x_d, y_d) \)
by \( \dot{r} = A_r r + B_r d \) where

\[
\begin{align*}
\begin{align*}
\hat{r} := \begin{pmatrix} x_r \\ y_r \end{pmatrix}, & \quad d := \begin{pmatrix} x_d \\ y_d \end{pmatrix}, & \quad A_r := \begin{pmatrix} -100 & 0 \\ 0 & -100 \end{pmatrix}, & \quad B_r := \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}.
\end{align*}
\end{align*}
\]

Fig. 5. Comparison of obtained maximum relative stability (left) and minimum disturbance attenuation level (right) under parameter-dependent disturbance \( \zeta(\theta) \).

Fig. 6. The structure for simulations of the controlled PVTOL aircraft.
In the actuator dynamics, the bandwidth for thrust \( u_1 \) is at 5 rad/s and the bandwidth for moment \( u_2 \) is at 14.28 rad/s. Therefore, the actuator dynamics is represented by

\[
A_a := \begin{pmatrix} -5 & 0 \\ 0 & -14.28 \end{pmatrix}, \quad B_a := \begin{pmatrix} 5 \\ 0 \end{pmatrix} 14.28.
\]

The sensor noises denoted by \( n_6 \) are described by band-limited white noises with sampling rate 0.1 s for all the six state variables. Then, the noises are weighted by \( W_n = \text{diag}(0.02, 0.03, 0.02, 0.03, 0.01, 0.02) \).

The parameter-dependent state-feedback matrix \( K(\theta) = L(Q_0 + p_sQ_s + p_cQ_c)^{-1} \) is obtained with the specified control effort magnitude constraints \( u_{1,\text{max}} = u_{2,\text{max}} = 0.6 \), varying parameter magnitude constraints \( \theta_{\text{max}} = \theta_{\text{max}} = \pi u_{2,\text{max}}/5 \), and disturbance attenuation performance level constraint \( \gamma_0 = 10 \). The simulations are performed and compared for three constructed controllers as follows:

- In the case that the disturbance \( \zeta(\theta) \) is simply considered as external input and the matrix inequality (28) is utilized in the optimization algorithm (33), the achieved maximum relative stability for the nominal design \( \varepsilon = 0 \) is \( z^* = -\mu^* = 0.0926 \) with matrix variable

\[
L = \begin{pmatrix} 0.0000 & -0.0000 & -0.0200 & -0.2105 & -0.0000 & 0.0000 \\ -0.0100 & -0.0366 & -0.0000 & 0.0000 & -0.0653 & -0.0090 \end{pmatrix}.
\] (40)

- In the case that the parameter-dependent characteristics of disturbance \( \zeta(\theta) \) described by the condition (11) are explicitly addressed and the matrix inequality (30) is used in the optimization algorithm (33), the achieved relative stability for the nominal design \( \varepsilon = 0 \) is \( z^* = -\mu^* = 0.1623 \) with matrix variable

\[
L = \begin{pmatrix} -0.0000 & 0.0000 & 0.0157 & -0.1242 & -0.0000 & 0.0000 \\ 0.0031 & -0.0493 & -0.0000 & -0.0000 & -0.0989 & -0.0232 \end{pmatrix},
\] (41)

and the achieved relative stability for the robust design \( \varepsilon = 1 \) is \( z^* = -\mu^* = 0.1501 \) with matrix variable

\[
L = \begin{pmatrix} 0.0001 & -0.0016 & 0.0042 & -0.1211 & -0.0007 & -0.0005 \\ -0.0573 & -0.0786 & 0.1033 & -0.0003 & -0.0888 & -0.0200 \end{pmatrix}.
\] (42)

Figs. 7 and 8 show the time responses of the closed-loop system with controllers constructed for nominal design with \( \varepsilon = 0 \). The tracking command is \((x_d, y_d) = (1, 1)\). The curves denoted with dotted lines are the responses for design with matrix variable in Eq. (40) and denoted with solid lines are responses for design with matrix variable in Eq. (41). It can be seen from Fig. 7 that the design considering the parameter-dependent characteristics of disturbance \( \zeta(\theta) \) can perform a faster position tracking responses, which agrees with the results of the obtained maximum relative stability \( z^* \) in the controller design phase. Also as shown in Fig. 8, it is observed that the faster position tracking corresponds to larger maneuvering on attitude \((\theta, \dot{\theta})\) and control moment \( u_2 \). Moreover, the quantities of the responses of \( \tilde{u}_1 = u_1 - 1, u_2, \theta \) and \( \dot{\theta} \) are well inside their specified magnitude constraints.

When a time-varying parasitic factor \( \varepsilon(t) = \sin 0.2t \) is present under the same command \((x_d, y_d) = (1, 1)\) for the nominal design with annihilation condition in Eq. (11) addressed
Fig. 7. Time response of \((x, y)\) and \(\epsilon\) for nominal design.

Fig. 8. Time response of \((\theta, \dot{\theta})\) and \((\tilde{u}_1, u_2)\) for nominal design.
and the matrix variable in Eq. (41). It is demonstrated that the non-minimum phase time delays are incurred in the response of lateral position $x$ as shown in Fig. 9, accompanied with larger magnitudes of attitude variation $(\theta, \dot{\theta})$ and control moment $u_2$, compared to the responses with $\varepsilon(t) = 0$ as shown in Fig. 8. However, if alternatively an exponential parasitic uncertainty $\varepsilon(t) = 1.5(1-e^{-0.2t})$ is included, as shown in the responses of Fig. 10 the resulting oscillations with increasing magnitudes reveal the instability of the closed-loop system.

In the case that an parasitic uncertainty $\varepsilon = 1$ is considered in the enhanced matrix inequality (30), the resulting matrix variable $L$ is shown in Eq. (42). The stability and tracking performance can be maintained as manifested from the time responses as shown in Fig. 11 even under the real-time parasitic uncertainty $\varepsilon(t) = 1.5(1-e^{-0.2t})$. The extra stability margin against the uncertainty factor $\varepsilon$ is expectable, since the synthesis matrix inequalities (28) or (30) are of sufficient conditions.

In the above discussions for various designs and the corresponding time responses of simulations in Figs. 7–11, the sensor noises $n\{6\}$ as shown in Fig. 6 are not imposed in order to clearly demonstrate the capabilities and limits of the design results. However, the sensor noises are inevitable in real-time applications. If the state measurements are corrupted with band-limited white noises weighted by $W_n$, the time responses of the system variables analogous to Fig. 11 are shown in Fig. 12 and the actual state measurements fed to control law $K(\theta)$ are shown in Fig. 13. Though the measurements are severely contaminated, the time responses are still satisfactory under the position tracking commands.

Fig. 9. Time response for nominal design with $\varepsilon = \sin 0.2t$. 
Fig. 10. Time response for nominal design with $\epsilon = 0.55(1-e^{-0.2t})$.

Fig. 11. Time response for robust design with $\epsilon = 1.5(1-e^{-0.2t})$. 
Fig. 12. Time response for robust design with $\varepsilon = 1.5(1-e^{-0.2t})$ and sensor noises.

Fig. 13. Time response of the contaminated state measurements.
5. Conclusion

This paper has presented a gain-scheduled control approach for the vertical takeoff and landing aircraft. The non-linear aircraft dynamics were formulated as a LPV system with external parameter-dependent disturbance, which arisen from the equilibrating between gravity force and nozzles thrust. The disturbance was shown to be dependent on the system varying parameter, roll angle, and a constant parameter denoting the normalized gravity force. The controllers were designed in terms of mixed optimization of $H_{\infty}$ performance for disturbance attenuation and relative stability for tracking position command in pitch-yaw plane. The characteristics of the parameter-dependent disturbance were described by an annihilation condition. The incorporation of the parameter-dependence condition on disturbance into the controller design algorithms based on linear matrix inequalities (LMIs) can deliver better performance than simply considering it as an external disturbance. The design results have been demonstrated by time response simulations. Also, the non-minimum phase effect caused by the undesired parasitic coupling between control moment and lateral force was addressed explicitly and verified in the simulations as well.

References