Electrostatic stress analysis of an anisotropic piezoelectric half-plane under surface electromechanical loading

J.Y. Liou a,*, J.C. Sung b

a Department of Civil Engineering, Kao Yuan University, Kaohsiung, Taiwan 82151, ROC
b Department of Civil Engineering, National Cheng Kung University, Tainan, Taiwan 70101, ROC

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Abstract

The generalized stress components on an anisotropic piezoelectric half-plane boundary under surface electromechanical loading are investigated. It is found that the behaviors of generalized stress components are related to matrices $\Gamma$ and $\Omega$, which have the same form as those for the purely elastostatic problem. Matrices $\Gamma$ and $\Omega$ contain all the electro-mechanical coupling phenomena of the generalized stress components. All elements of matrices $\Gamma$ and $\Omega$ are expressed explicitly in terms of generalized elastic stiffness for monoclinic piezoelectric materials with the plane of symmetry at $x_3 = 0$ and for transversely isotropic piezoelectric materials in which the coupled effects between the mechanical (electrical) deformations induced by electrical (mechanical) loadings are studied analytically. A numerical example of the electro-mechanical coupling behavior for PZT-4 is also given.

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1. Introduction

Electro-mechanical coupling effects of piezoelectric materials have received much attention because they have a wide range of applications. Cracks occur frequently in piezoelectric materials. Accordingly, many researchers, such as Pak (1990), Sosa and Pak (1990), Park and Sun (1995), Kuang et al. (2004), and others, have for a long time examined electro-mechanical fields at the crack tips. The coupling effects of piezoelectric materials with holes or inclusions are also important and many researchers, such as Zhang et al. (1998), Xu and Rajapakse (2001), Dai et al. (2006) have paid attention to them. This work examines different electro-mechanical coupling effects, i.e., explores the phenomena of the electro-mechanical coupled generalized stress...
components on the boundary induced by surface electromechanical loading that is applied to the straight boundary of anisotropic piezoelectric half-plane solids.

For a half-plane anisotropic piezoelectric solid whose straight boundary is subjected to a uniform generalized traction $\mathbf{t}_2 = [\tau, \sigma, \mathbf{r}, D]^{T}$ (Fig. 1) with a finite loaded region, the electro-mechanical coupled generalized stress components induced on the boundary may be denoted by $\mathbf{t}_1 = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_{1}]^{T}$. Since $\sigma_{12}$ is completely determined by the boundary conditions, only the behaviors of the mechanical stress components $\sigma_{11}$ and $\sigma_{13}$ coupled with the component of electrical displacement $D_{1}$ are of interests in our analyses. Similar problem but for anisotropic elastic materials has been investigated by Liou and Sung (2008). The behaviors of the coupled generalized stress components for anisotropic piezoelectric material are similar to those for purely anisotropic materials (Liou and Sung, 2008), i.e., the coupled generalized stress components consist of generally a constant term and a logarithmic term. The constant term appears only in the loaded region and the logarithmic term appears in both the loaded and unloaded regions. This correspondence between plane piezoelectricity and generalized plane strain in elasticity has been noted by Chen and Lai (1997). The constant term is determined by the matrix $\Omega$ and the logarithmic term is determined by the matrix $\Gamma$. Elements of matrices $\Omega$ and $\Gamma$ contain all the electro-mechanical coupled information. Those elements of $\Omega$ and $\Gamma$ which correspond to mechanical deformations induced by mechanical loadings are similar to those explored by Liou and Sung (2008) for purely elastic problems. Therefore, present investigations focus only on those elements which correspond to mechanical (electrical) deformations induced by electrical (mechanical) loadings. Explicit expressions for elements of $\Omega$ and $\Gamma$ in terms of generalized elastic stiffness are obtained for monoclinic piezoelectric material with the plane of symmetry at $x_{3} = 0$ and for transversely isotropic piezoelectric material. With obtained analytic expressions for both materials, the effects of material constants on the coupled mechanical and electric phenomena are further explored. By taking special values for the elements of $\Omega$ and $\Gamma$, the results for decoupled mechanical deformations and electrical field are both recovered. The coupling behaviors for the elements of $\Omega$ and $\Gamma$ for the material PZT-4 are studied numerically as an example.

2. Generalized Stroh formalism

Before presenting the coupled generalized stress components for anisotropic piezoelectric materials, we briefly introduce the generalized Stroh formalism in this section. The conventions that all Latin indices range from 1 to 3 (except where indicated) and that repeated indices imply summation are all followed. Bold-faced symbol stands for either column vectors or matrices depending on whether lower-case or upper-case is used. For two-dimensional anisotropic piezoelectric deformations, the generalized displacement vector
\[ \mathbf{u} = [u_1, u_2, u_3, u_4]^T \text{ (} u_i, i = 1, 2, 3; \text{ the elastic displacements; } u_4; \text{ electric potential) and the generalized stress function vector, } \phi = [\phi_1, \phi_2, \phi_3, \phi_4]^T \text{ are expressed as} \]
\[ \begin{align*}
\mathbf{u} &= 2\text{Re}\{\mathbf{A}f(z)\}, \\
\phi &= 2\text{Re}\{\mathbf{B}f(z)\},
\end{align*} \tag{2.1} \]
where
\[ \mathbf{A} = [a_1, a_2, a_3, a_4], \tag{2.3} \]
\[ \mathbf{B} = [b_1, b_2, b_3, b_4], \tag{2.4} \]
\[ f(z) = [f_1(z), f_2(z), f_3(z), f_4(z)]^T, \tag{2.5} \]
where the superscript T indicates the transpose and \( z_k = x_1 + p_k x_2 \). Unknown complex number \( p_k \) and constant vector \( a_k \) are determined by the eigenrelation
\[ [\mathbf{Q} + p_k (\mathbf{R} + \mathbf{R}^T) + p_k^2 \mathbf{T}]a_k = 0, \quad (k = 1, 2, 3, 4) \tag{2.6} \]
where
\[ \mathbf{Q} = \begin{bmatrix} c_{1111} & e_{111} \\ e_{111} & -x_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} c_{1122} & e_{112} \\ e_{212} & -x_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{2222} & e_{222} \\ e_{221} & -x_{22} \end{bmatrix}. \tag{2.7} \]

The material constants defined in the matrices \( \mathbf{Q} \), \( \mathbf{R} \), and \( \mathbf{T} \) originate from the constitutive equations specified for a linear piezoelectric material (Ting, 1996), which are given by
\[ \sigma_{ij} = c_{ijkl} e_{ks} - e_{ij} E_k, \quad D_i = e_{ik} e_{ks} + x_{ik} E_k, \tag{2.8} \]
where \( e_{ij} \) and \( E_k \) are the strains and electric fields, \( \sigma_{ij} \) and \( D_i \) are the stresses and electrical displacements, and \( c_{ijkl} \) and \( x_{ij} \) are the elastic stiffness, piezoelectric-stress and dielectric constants, respectively. The generalized stress function vector expressed in Eq. (2.2) may be directly employed to evaluate the stress and electric displacement components, using the formula,
\[ \mathbf{t}_1 = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1]^T = -\frac{\partial \phi}{\partial x_1}, \quad \mathbf{t}_2 = [\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2]^T = \frac{\partial \phi}{\partial x_1}. \tag{2.9} \]

Note that the column vectors of matrix \( \mathbf{B} = [b_1, b_2, b_3, b_4] \) that appear in Eq. (2.4) are related to the column vectors of matrix \( \mathbf{A} = [a_1, a_2, a_3, a_4] \) in the following form
\[ b_k = (\mathbf{R}^T + p_k \mathbf{T})a_k, \quad k = 1, 2, 3, 4. \tag{2.10} \]

Using orthogonality relationship (Ting, 1996), matrices \( \mathbf{A} \) and \( \mathbf{B} \) defined above may be employed to define the three real matrices \( \mathbf{L}, \mathbf{S}, \) and \( \mathbf{H} \), called the generalized Barnett–Lothe tensors (Ting, 1996), as follows
\[ \mathbf{L} = -2i \mathbf{B} \mathbf{B}^T, \quad \mathbf{S} = i(2\mathbf{A} \mathbf{B}^T - \mathbf{I}), \quad \mathbf{H} = 2i \mathbf{A} \mathbf{A}^T, \tag{2.11} \]
where \( i^2 = -1 \) and \( \mathbf{I} \) is a \( 4 \times 4 \) unit real matrix.

### 3. Coupled generalized stress components for anisotropic piezoelectric materials

The electro-mechanical coupled generalized stress components induced by finite extended uniform generalized traction over the half-plane boundary of an anisotropic piezoelectric material are presented in this section. Analyses of this problem parallel those developed for purely anisotropic elastic materials (Liou and Sung, 2008). Firstly, note that the responses of a half-plane piezoelectric solid subjected to a generalized concentrated force \( \mathbf{f} \) at the origin are as follows (Ting, 1996)
\[ \begin{align*}
\mathbf{u} &= [u_1, u_2, u_3, u_4]^T = -\left[ \frac{1}{\pi} (\ln r) \mathbf{I} + \mathbf{S}(0) \right] \mathbf{L}^{-1} \mathbf{f}, \\
\phi &= [\phi_1, \phi_2, \phi_3, \phi_4]^T = \mathbf{L}(0) \mathbf{L}^{-1} \mathbf{f},
\end{align*} \tag{3.1} \]
where
\[ S(\theta) = \frac{1}{\pi} \int_{0}^{\theta} N_{1}(\omega) d\omega, \quad L(\theta) = \frac{1}{\pi} \int_{0}^{\theta} N_{3}(\omega) d\omega, \quad (3.2a) \]

\[ N_{1}(\omega) = -T^{-1}(\omega)R^{T}(\omega), \quad N_{3}(\omega) = R(\omega)T^{-1}(\omega)R^{T}(\omega) - Q(\omega), \quad (3.2b) \]

\[ Q(\omega) = \begin{bmatrix} Q_{11}(\omega) & e_{11}(\omega) \\ e_{11}^{T}(\omega) & -\alpha_{11}(\omega) \end{bmatrix}, \quad R(\omega) = \begin{bmatrix} R_{1}(\omega) & e_{21}(\omega) \\ e_{21}^{T}(\omega) & -\alpha_{21}(\omega) \end{bmatrix}, \quad T(\omega) = \begin{bmatrix} T_{1}(\omega) & e_{22}(\omega) \\ e_{22}^{T}(\omega) & -\alpha_{22}(\omega) \end{bmatrix}, \quad (3.2c) \]

\[ Q_{ik}^{E}(\omega) = c_{ijk}n_{j}n_{j}, \quad R_{ik}^{E}(\omega) = c_{ijk}n_{j}m_{j}, \quad T_{ik}^{E}(\omega) = c_{ijk}m_{j}m_{j}, \quad (3.2d) \]

\[ e_{11}(\omega) = e_{ij}n_{i}n_{j}, \quad e_{12}(\omega) = e_{ij}n_{i}m_{j}, \quad e_{21}(\omega) = e_{ij}m_{i}n_{j}, \quad e_{22}(\omega) = e_{ij}m_{i}m_{j}, \quad (3.2e) \]

\[ \alpha_{11}(\omega) = \alpha_{ij}n_{j}n_{j}, \quad \alpha_{12}(\omega) = \alpha_{ij}n_{j}m_{j}, \quad \alpha_{22}(\omega) = \alpha_{ij}m_{j}m_{j}. \quad (3.2f) \]

\[ n = [\cos \omega, \sin \omega, 0]^{T}, \quad m = [-\sin \omega, \cos \omega, 0]^{T}. \quad (3.2g) \]

With \( \omega = 0 \) in Eq. (3.2b), the results become

\[ N_{1} = -T^{-1}R^{T}, \quad N_{3} = RT^{-1}R^{T} - Q, \quad (3.3) \]

where \( N_{1} = N_{1}(0) \) and \( N_{3} = N_{3}(0) \) are the sub-matrices defined in the matrix \( N \), known as the generalized fundamental elasticity matrix (Ting, 1996). Using the results in Eq. (3.1), the responses induced by finite extended uniform generalized traction over the half-plane boundary may be evaluated simply by integration, with \( f \) in Eq. (3.1) replaced by \( f \ dx = [\tau, \sigma, \tilde{D}]^{T} \ dx \), where \( f = [\tau, \sigma, \tilde{D}]^{T} \) is now interpreted as the intensity of the uniform generalized tractions. The whole procedure for the calculation of the generalized stress components parallels that for purely anisotropic elastic materials (Liou and Sung, 2008). Moreover, the obtained results for the generalized stress components are exactly in the same forms as those for purely anisotropic elastic materials (Liou and Sung, 2008), i.e., for the region outside the loading part, \( |x_{1}| > a \), the coupled generalized stress components are

\[ t_{i}(x_{1}, 0) = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_{1}]^{T} = \frac{-1}{\pi} \ln \left( \frac{|x_{1} - a|}{|x_{1} + a|} \right) \Gamma f, \quad (3.4) \]

where

\[ \Gamma = N_{3}L^{-1}, \quad (3.5) \]

and for the loading area, \( |x_{1}| < a \), the coupled generalized stress components are

\[ t_{i}(x_{1}, 0) = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_{1}]^{T} = \frac{-1}{\pi} \ln \left( \frac{a - x_{1}}{a + x_{1}} \right) \Omega f - \Omega f, \quad (3.6) \]

where

\[ \Omega = (N_{3}S - N_{3}^{T}L)L^{-1}. \quad (3.7) \]

The correspondence between plane piezoelectricity and generalized plane strain in elasticity has been noted by Chen and Lai (1997). Note also that similar results shown in Eqs. (3.4) and (3.6) have been obtained by Fan et al. (1996) where the problem of stress and electrical field distributions in a piezoelectric half-plane under contact load is investigated. Even though Eqs. (3.4) and (3.6) take the same forms as those for purely anisotropic elastic materials, there are differences between piezoelectric and purely elastic materials which arise from the elements of the matrices \( \Omega \) and \( \Gamma \). It can be shown that the explicit structures of \( \Omega \) and \( \Gamma \) take the following form (Liou and Sung, 2008):

\[ \Gamma = (N_{3}L)^{-1} = \begin{bmatrix} \Gamma_{xx} & \Gamma_{xn} & \Gamma_{nx} & \Gamma_{ne} \\ 0 & 0 & 0 & 0 \\ \Gamma_{ex} & \Gamma_{en} & \Gamma_{ee} & \Gamma_{ve} \\ \Gamma_{es} & \Gamma_{en} & \Gamma_{ev} & \Gamma_{ve} \end{bmatrix}, \quad \Omega = (N_{3}S - N_{3}^{T}L)^{-1} = \begin{bmatrix} \Omega_{xx} & \Omega_{xn} & \Omega_{nx} & \Omega_{ne} \\ 1 & 0 & 0 & 0 \\ \Omega_{ex} & \Omega_{en} & \Omega_{ee} & \Omega_{ve} \\ \Omega_{es} & \Omega_{en} & \Omega_{ev} & \Omega_{ve} \end{bmatrix}. \quad (3.8) \]
Note that the meanings of the sub-indices for the elements of matrices $\Gamma$ and $\Omega$ are similar to those explained by Liou and Sung (2008). For instance, the first subscript $n$ of $\Gamma_{ne}$ denotes that the coupled generalized stress components induced on the surface is $\sigma_{11}$ while the second subscript $e$ of $\Gamma_{ne}$ denotes that the type of loading applied to the surface is electric displacement loading. The nonzero elements of matrices $\Gamma$ and $\Omega$ given above, contain all information of the coupling effects of the mechanical deformations coupled with the electric fields for general anisotropic piezoelectric materials. Elements of $\Gamma$ and $\Omega$ which correspond to the mechanical deformations induced by mechanical loadings (involving the three sub-indices $n$, $s$ and $v$) are similar to those explored by Liou and Sung (2008) for purely elastic deformations. Therefore, present investigations focus only on those elements of $\Gamma$ and $\Omega$ that account for the coupled phenomena of mechanical deformations (electrical field) induced by electrical (mechanical) loadings. In the next two sections, the coupled effects are further elucidated for monoclinic piezoelectric materials and transversely isotropic piezoelectric materials. Before leaving this section, we note that with elements of matrices $\Gamma$ and $\Omega$ shown in Eq. (3.8), the coupled generalized stress components given in Eqs. (3.4) and (3.6) may be written together as follows

$$
\mathbf{t}_1(x_1, 0) = \begin{bmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{13} \\
D_1
\end{bmatrix} = -\frac{1}{\pi} \ln \left( \frac{|x_1 - a|}{|x_1 + a|} \right) \begin{bmatrix}
\Gamma_{nn} \\
\Gamma_{vn} \\
\Gamma_{en} \\
\Gamma_{ee}
\end{bmatrix} + \tau \begin{bmatrix}
0 \\
0 \\
\Gamma_{ex} \\
\Gamma_{ee}
\end{bmatrix} + \tilde{\tau} \begin{bmatrix}
0 \\
0 \\
\Gamma_{ev} \\
\Gamma_{ev}
\end{bmatrix} + \tilde{D} \begin{bmatrix}
\Gamma_{ne} \\
0 \\
0 \\
\Gamma_{ee}
\end{bmatrix}
\}
(3.9)

where $H(x_1)$ is the Heaviside function. Note also that when the piezoelectric-stress constants vanish, i.e., $e_{ijk} = 0$, the elements of $\Gamma$ and $\Omega$ that correspond to the purely elastic part are recovered (Liou and Sung, 2008).

4. Monoclinic piezoelectric material with a plane of symmetry at $x_3 = 0$

In this section, the coupled generalized stress components on the boundary for monoclinic piezoelectric materials are addressed. To explore the behaviors of these coupled components, the explicit expressions for the elements of $\Gamma$ and $\Omega$ for monoclinic piezoelectric materials are needed which are discussed below. For monoclinic piezoelectric materials of class m, with a symmetry plane at $x_3 = 0$ (Nye, 1985), the corresponding constitutive equation (Eq. (2.8)) is

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{32} \\
\sigma_{31} \\
D_1 \\
D_2 \\
\{3\}
\end{bmatrix} = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} & e_{11} & e_{21} & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} & e_{12} & e_{22} & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} & e_{13} & e_{23} & 0 \\
0 & 0 & 0 & c_{44} & c_{45} & 0 & 0 & 0 & e_{34} \\
0 & 0 & 0 & c_{45} & c_{55} & 0 & 0 & 0 & e_{35} \\
0 & 0 & 0 & c_{66} & e_{16} & e_{26} & 0 & 0 & 0 \\
e_{11} & e_{12} & e_{13} & 0 & 0 & e_{16} & -x_{11} & -x_{12} & 0 \\
e_{21} & e_{22} & e_{23} & 0 & 0 & e_{26} & -x_{21} & -x_{22} & 0 \\
e_{34} & e_{35} & e_{36} & 0 & 0 & 0 & 0 & 0 & -x_{33}
\end{bmatrix} \begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
e_{34} \\
e_{35} \\
e_{36} \\
e_{11} \\
e_{12} \\
e_{13} \\
e_{16} \\
e_{21} \\
e_{22} \\
e_{23} \\
e_{26} \\
e_{34} \\
e_{35} \\
e_{36}
\end{bmatrix} = \begin{bmatrix}
e_{11} \\
e_{12} \\
e_{13} \\
e_{16} \\
e_{21} \\
e_{22} \\
e_{23} \\
e_{26} \\
e_{34} \\
e_{35} \\
e_{36}
\end{bmatrix}.
(4.1)

Where contracted notations $e_{\alpha\beta}$ and $c_{\alpha\beta\gamma}(\alpha, \beta = 1, 2, \ldots, 6)$ have been used here for piezoelectric-stress constants $e_{ijk}$ and elastic stiffness $c_{ijk\delta}$, respectively. For the material shown above, the explicit expressions of the matrices $Q$, $R$, and $T$ are as follows:
The needed expressions have been given by Liou and Sung (2007) for matrices $L^{-1}$ and $SL^{-1}$ and the results are as follows:

$$L^{-1} = \begin{bmatrix} Y_{11} & Y_{12} & 0 & Y_{14} \\ Y_{12} & Y_{22} & 0 & Y_{24} \\ 0 & 0 & Y_{33} & 0 \\ Y_{14} & Y_{24} & 0 & Y_{44} \end{bmatrix}, \quad SL^{-1} = \begin{bmatrix} 0 & -\dot{Y}_{12} & 0 & -\dot{Y}_{14} \\ \dot{Y}_{12} & 0 & 0 & -\dot{Y}_{24} \\ 0 & 0 & 0 & 0 \\ \dot{Y}_{14} & \dot{Y}_{24} & 0 & 0 \end{bmatrix},$$

(4.3)

where explicit expressions of all nonzero elements $Y_{ij}$ and $\dot{Y}_{ij}$ are given in Appendix A. Now, substituting all relevant matrices into Eqs. (3.8) and (3.10) yields the explicit structures of matrices $\Gamma$ and $\Omega$ as

$$\Gamma = (N_3L^{-1}) = \begin{bmatrix} \Gamma_{ns} & \Gamma_{nn} & 0 & \Gamma_{ne} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{ex} & 0 \\ \Gamma_{es} & \Gamma_{en} & 0 & \Gamma_{ee} \end{bmatrix}, \quad \Omega = (N_3SL^{-1} - N_1^T) = \begin{bmatrix} \Omega_{ns} & \Omega_{en} & 0 & \Omega_{ne} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{ex} & 0 \\ \Omega_{es} & \Omega_{en} & 0 & \Omega_{ee} \end{bmatrix},$$

(4.4)

where all nonzero elements of matrices $\Gamma$ and $\Omega$, expressed in terms of generalized elastic stiffness, are

$$\Gamma_{ns} = Y_{11}\left(\frac{G_1}{D_T} - c_{11}\right) + Y_{14}\left(\frac{G_2}{D_T} - e_{11}\right), \quad \Gamma_{nn} = Y_{12}\left(\frac{G_1}{D_T} - c_{11}\right) + Y_{24}\left(\frac{G_2}{D_T} - e_{11}\right),$$

(4.5)

$$\Gamma_{ne} = Y_{14}\left(\frac{G_1}{D_T} - c_{11}\right) + Y_{44}\left(\frac{G_2}{D_T} - e_{11}\right), \quad \Gamma_{ex} = Y_{11}\left(\frac{G_2}{D_T} - e_{11}\right) + Y_{14}\left(\frac{G_3}{D_T} + \alpha_{11}\right),$$

$$\Gamma_{en} = Y_{12}\left(\frac{G_2}{D_T} - e_{11}\right) + Y_{24}\left(\frac{G_3}{D_T} + \alpha_{11}\right), \quad \Gamma_{ee} = Y_{14}\left(\frac{G_2}{D_T} - e_{11}\right) + Y_{44}\left(\frac{G_3}{D_T} + \alpha_{11}\right),$$

$$\Gamma_{es} = Y_{33}\frac{c_{44}^2 - c_{45}^2}{c_{44}} = -\sqrt{c_{44}c_{55} - c_{45}^2} \frac{c_{44}}{c_{44}},$$

$$\Omega_{ns} = \dot{Y}_{14}\left(\frac{G_2}{D_T} - e_{11}\right) + \frac{W_{11}}{D_T}, \quad \Omega_{en} = \dot{Y}_{14}\left(\frac{G_1}{D_T} - c_{11}\right) + \frac{\dot{Y}_{24}}{D_T},$$

(4.6)

$$\Omega_{ne} = \dot{Y}_{14}\left(\frac{G_1}{D_T} - c_{11}\right) + \frac{W_{14}}{D_T}, \quad \Omega_{ee} = -\dot{Y}_{14}\left(\frac{G_3}{D_T} + \alpha_{11}\right) + \frac{W_{41}}{D_T},$$

$$\Omega_{es} = -\dot{Y}_{12}\left(\frac{G_2}{D_T} - e_{11}\right) + \dot{Y}_{24}\left(\frac{G_3}{D_T} + \alpha_{11}\right) + \frac{W_{42}}{D_T}, \quad \Omega_{ce} = -\dot{Y}_{14}\left(\frac{G_2}{D_T} - e_{11}\right) + \frac{W_{44}}{D_T},$$

and

$$D_T = c_{66}e_{22}\alpha_{22} - 2c_{26}e_{22}e_{26} + c_{22}e_{26}^2 - \alpha_{22}c_{26}^2 + c_{66}e_{22}^2,$$
\[ \mathbf{\Omega} \] and \[ \mathbf{\Omega} \] given in Eq. (4.4), the generalized stress components on the boundary may be expressed as

\[
\mathbf{t}_1(x_1, 0) = \begin{pmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{13}
\end{pmatrix} = \frac{-1}{\pi} \ln \left( \frac{|x_1-a|}{|x_1+a|} \right) \begin{bmatrix}
\Gamma_{nn} \\
\Gamma_{ne} \\
\Gamma_{ee}
\end{bmatrix} + \begin{bmatrix}
\Gamma_{nn} \\
\Gamma_{ne} \\
\Gamma_{ee}
\end{bmatrix} + \tilde{\mathbf{D}} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + \tilde{\mathbf{D}} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\Omega_{nn} \\
\Omega_{ne} \\
\Omega_{ee}
\end{bmatrix} H(a - |x_1|),
\]

(4.7)

where all nonzero elements of matrices \( \mathbf{\Gamma} \) and \( \mathbf{\Omega} \) are given by Eq. (4.5). Therefore, from Eq. (4.7), the following observations of the electro-mechanical coupled generalized stress components for monoclinic materials with a symmetry plane at \( x_3 = 0 \), can be made:

1. Elements \( \Gamma_{ee} = \Omega_{ee} = 0 \) indicate no response of the electrical field when anti-plane shear loading is applied. Similarly, the elements of \( \Gamma_{ee} = \Omega_{ee} = 0 \) imply that no anti-plane shear deformation will be induced by electric loading. Thus, no coupled effect occurs between the anti-plane mechanical deformation and the electrical field for monoclinic materials with a symmetry plane at \( x_3 = 0 \).

2. Coupled effects do occur between in-plane mechanical deformation and electrical field since (i) elements \( \Gamma_{en} \neq 0 \) and \( \Omega_{en} \neq 0 \), indicating that logarithmic and constant terms exist for the electric field when in-plane normal loading is applied. (ii) Similarly, since \( \Gamma_{es} \neq 0 \) and \( \Omega_{es} \neq 0 \), the electrical field exists when in-plane shear loading is applied. (iii) When electrical loading is applied, the coupled effects between in-plane mechanical deformation and the electrical field can be observed since the nonzero elements of \( \Gamma_{ne} \) and \( \Omega_{ne} \) imply that logarithmic and constant distributions exist for the mechanical stress component \( \sigma_{11} \). Notably, \( \Gamma_{se} = 0 \) and \( \Omega_{se} = 0 \), indicate that when only electric loading is applied, no mechanical stress component \( \sigma_{12} \) is induced on the boundary. This is the boundary condition that is specified for \( \sigma_{12} \) which should be satisfied.

5. Transversely isotropic piezoelectric materials

The discussions of coupled generalized stress components for transversely isotropic piezoelectric materials parallel those given in the previous section. For transversely isotropic piezoelectric materials whose \( x_2 \)-axis is
parallel to the poling direction, the constitutive equation can be obtained from monoclinic piezoelectric materials by letting $c_{45} = c_{16} = c_{26} = c_{36} = 0, c_{23} = c_{12}, c_{33} = c_{11}, c_{66} = c_{44}, c_{55} = (c_{11} - c_{13})/2$, $e_{11} = e_{12} = e_{13} = e_{26} = e_{35} = e_{12} = 0, e_{23} = e_{21}$, and $e_{34} = e_{16}$. Therefore, matrices $Q, R$, and $T$ defined in Eq. (2.7) now become

$$Q = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{44} & 0 & e_{16} \\ 0 & 0 & c_{55} & 0 \\ 0 & e_{16} & 0 & -\varepsilon_{11} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & c_{12} & 0 & e_{16} \\ c_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_{21} & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} c_{44} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & e_{22} \\ 0 & 0 & c_{44} & 0 \\ 0 & e_{22} & 0 & -\varepsilon_{22} \end{bmatrix},$$

and matrices $L^{-1}$ and $SL^{-1}$ for transversely isotropic piezoelectric materials become

$$L^{-1} = \begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & Y_{24} \\ 0 & 0 & Y_{33} & 0 \\ 0 & Y_{24} & 0 & Y_{44} \end{bmatrix}, \quad SL^{-1} = \begin{bmatrix} 0 & -\tilde{Y}_{12} & 0 & -\tilde{Y}_{14} \\ \tilde{Y}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{Y}_{14} & 0 & 0 & 0 \end{bmatrix},$$

where nonzero elements of $Y_{\alpha\beta}$ and $\tilde{Y}_{\alpha\beta}$ may be obtained from Appendix A with material properties appropriate for transversely isotropic piezoelectric materials being used. Similar to the procedures described in Section 4, the explicit structures of $\Gamma$ and $\Omega$ may be obtained as follows:

$$\Gamma = (N_3L^{-1}) = \begin{bmatrix} \Gamma_{ns} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{ve} & 0 \\ 0 & \Gamma_{en} & 0 & \Gamma_{ee} \end{bmatrix}, \quad \Omega = (N_3SL^{-1} - N_1) = \begin{bmatrix} 0 & \Omega_{mn} & 0 & \Omega_{ne} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Omega_{es} & 0 & 0 & 0 \end{bmatrix},$$

where nonzero elements, expressed in terms of generalized elastic stiffness, are as follows

$$\Gamma_{ns} = \frac{Y_{11}(2c_{12}e_{22}e_{21} - c_{11}c_{22}e_{22} - c_{11}e_{2}^2 + c_{22}e_{21}^2 - c_{22}e_{21}^2)}{c_{22}e_{22} + e_{22}^2},$$

$$\Gamma_{ee} = -c_{55}Y_{33} = \sqrt{\frac{c_{55}}{c_{44}}}, \quad \Gamma_{en} = Y_{24}(e_{16}^2 + x_{11}e_{44})/c_{44}, \quad \Gamma_{ee} = Y_{44}(e_{16}^2 + x_{11}e_{44})/c_{44},$$

$$\Omega_{mn} = \frac{-\tilde{Y}_{12}(2c_{12}e_{22}e_{21} - c_{11}c_{22}e_{22} - c_{11}e_{2}^2 + c_{22}e_{21}^2 + c_{22}e_{21}^2)}{c_{22}e_{22} + e_{22}^2},$$

$$\Omega_{ne} = \frac{-\tilde{Y}_{14}(2c_{12}e_{22}e_{21} - c_{11}c_{22}e_{22} - c_{11}e_{2}^2 + c_{22}e_{21}^2 + c_{22}e_{21}^2)}{c_{22}e_{22} + e_{22}^2},$$

$$\Omega_{es} = [\tilde{Y}_{14}(e_{16}^2 + x_{11}e_{44}) + e_{16}]/c_{44}.$$

With the explicit structures of $\Gamma$ and $\Omega$ given in Eq. (5.3), the coupled generalized stress components on the boundary may be expressed as

$$t_1(x_1, 0) = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ D_1 \end{bmatrix} = \frac{1}{\pi} \ln \left( \frac{|x_1 - a|}{|x_1 + a|} \right) \begin{bmatrix} \sigma \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \Gamma_{ns} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \hat{T} \begin{bmatrix} 0 \\ 0 \\ \Gamma_{ee} \\ \Gamma_{ee} \end{bmatrix} + \hat{D} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + H(a - |x_1|),$$

(5.5)
where $\Gamma_{en}$, $\Gamma_{ns}$, $\Gamma_{ev}$, $\Gamma_{ee}$, $Q_{ms}$, $Q_{es}$ and $\Omega_{ne}$ are given by Eq. (5.4). Therefore, from Eq. (5.5), the following observations of the electro-mechanical coupled generalized stress components for transversely isotropic piezoelectric materials with the $x_2$-axis parallel to the poling direction can be made:

1. No coupled effect exists between anti-plane mechanical deformation and electrical field since $\Gamma_{ev} = -\Omega_{ee} = 0$ and $\Gamma_{ne} = \Omega_{en} = 0$.
2. Coupled effects do occur between in-plane mechanical deformation and the electrical field. However, the coupled phenomena differ from those discussed for monoclinic materials. (i) Consider first the application of mechanical normal loading. For monoclinic materials, recall that $\Gamma_{en} \neq 0$ and $Q_{en} \neq 0$, as discussed previously. For transversely isotropic material, however, $\Gamma_{en} = 0$ and $Q_{en} = 0$, indicating that under mechanical normal loading, no constant term is induced for the electric field; instead, a logarithmic term exists for the electric field. (ii) Secondly, consider the application of mechanical shear loading. The observed phenomenon is different from that under normal loading, i.e., the induced electric field has a constant term instead of a logarithmic term since $\Gamma_{es} = 0$ and $Q_{es} \neq 0$. (iii) Finally, consider the application of electrical loading. The appearance of the terms $\Gamma_{ne} = 0$ and $\Omega_{ne} \neq 0$ implies that only a constant distribution for the mechanical stress $\sigma_{11}$ on the boundary is induced by the electrical loadings. Note that the elements that correspond to the responses of the mechanical stress $\sigma_{12}$ induced by the electrical loadings are both zeros, i.e., $\Gamma_{se} = 0$ and $\Omega_{se} = 0$, implying that no mechanical stress $\sigma_{12}$ occurs when only electric loadings is applied. This is again the boundary condition specified for $\sigma_{12}$.

Note that if the poling direction of the transversely isotropic piezoelectric materials is initially along the $x_1$-axis, and not the $x_2$-axis, as discussed above, it can be shown that the corresponding coupled generalized stress components on the boundary are as follows

$$
t_1(x_1, 0) = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{bmatrix} = \frac{-1}{\pi} \ln \left( \frac{|x_1|}{|x_1|+a} \right) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \hat{\tau} \begin{bmatrix} \Gamma_{ns} \\ \Gamma_{ev} \\ \Gamma_{ee} \end{bmatrix} + \hat{D} \begin{bmatrix} \Omega_{en} \\ \Omega_{es} \\ \Omega_{ee} \end{bmatrix} \right) \right) \right) H(a - |x_1|),
$$

which differ slightly from those given in Eq. (5.5).

For completeness of the discussion of coupled electro-mechanical effects, consider the simplest case, i.e., consider the case for isotropic materials. For isotropic materials, the material constants can be further simplified to

$$
c_{16} = c_{26} = c_{45} = 0, \quad e_{11} = e_{12} = e_{26} = \epsilon_{12} = 0, \quad \epsilon_{22} = \epsilon_{11},
$$

$$
c_{66} = c_{55} = c_{44} = \mu, \quad c_{22} = c_{11} = \lambda + 2\mu, \quad c_{12} = \lambda,
$$

where $\lambda$ and $\mu$ are Lamé’ constants and the corresponding matrices for $L^{-1}$ and $SL^{-1}$ are

$$
L^{-1} = \begin{bmatrix} Y_{11} & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & 0 \\ 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & Y_{44} \end{bmatrix}, \quad SL^{-1} = \begin{bmatrix} 0 & -\tilde{Y}_{12} & 0 & 0 \\ \tilde{Y}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

where

$$
Y_{11} = Y_{22} = \frac{1 - \nu}{\mu}, \quad Y_{33} = \frac{1}{\mu}, \quad Y_{44} = \frac{-1}{\epsilon_{11}}, \quad \tilde{Y}_{12} = \frac{1 - 2\nu}{2\mu}.
$$
One may easily verify that for isotropic material matrices, $\Gamma$ and $\Omega$ become

$$
\Gamma = (N_3 L^{-1}) = \begin{bmatrix}
-2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}, \quad \Omega = (N_3 S L^{-1} - N_1^T) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

which show that all of the elements of $\Gamma$ and $\Omega$ are material-independent for isotropic material matrices. Results in Eq. (5.10) also indicate that all of the elements related to mechanical and electric coupled deformations all vanish, i.e.,

$$
\Gamma_{en} = \Gamma_{er} = \Gamma_{ec} = \Gamma_{ce} = \Gamma_{se} = \Gamma_{sc} = 0, \quad \Omega_{en} = \Omega_{er} = \Omega_{ec} = \Omega_{ce} = \Omega_{se} = \Omega_{sc} = 0.
$$

Therefore, no mechanical and electric coupling effect is observed for isotropic materials. The coupled generalized stress components on the boundary are totally decoupled into two parts: one corresponds to the responses of purely isotropic elastic problems and the other corresponds to the responses of purely electric problems, and they are mutually independent. The results for the decoupled generalized stress components are

$$
t_1(x_1, 0) = \frac{1}{3} \ln \left[ \frac{\left| x_1 - a \right|}{\left| x_1 + a \right|} \right] \begin{bmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{13} \\
D_1 \\
\end{bmatrix} = \begin{bmatrix}
0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix} \begin{bmatrix}
\sigma \\
\tau \\
\hat{\sigma} \\
\hat{\tau} \\
\end{bmatrix} + \hat{D} \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix} \left| x_1 \right|
$$

revealing that the well-known results for isotropic elastic deformation (England, 1971; Muskhelishvili, 1953) and the electrical field are both recovered.

### 6. Numerical results and discussion

The electro-mechanical coupled generalized stress components on the boundary studied in previous sections are for cases in which the material’s principal axes coincide with the coordinates. This section investigates the effects of the material’s alignment on the coupled components on the boundary. Let us take the materials discussed in Section 5. Take PZT-4, which is a transversely isotropic piezoelectric material with the $x_3$-axis parallel to the poling direction (Appendix B presents the material constants of this piezoelectric ceramic. It also presents the matrices $L^{-1}$ and $SL^{-1}$ needed in the computations of the elements of matrices $\Gamma$ and $\Omega$). Suppose that the material’s principal axes, which initially coincide with the coordinate system, are now rotated clockwise by an angle $\gamma$ about the $x_3$-axis, new material’s constants are then obtained with respect to the original coordinate system, which belongs to the class of monoclinic materials. Note that the material alignment with $\gamma = 90^\circ$ is also a transversely isotropic piezoelectric material but with the poling direction now parallel to the $x_1$-axis. Figs. 2–4 plot the effects of the material’s alignment on the elements of $\Gamma$ and $\Omega$. Only the elements of $\Gamma_{2\beta}$ and $\Omega_{2\beta}$ that are not zeros are presented (see Eq. (4.4)). In these Figures, both coupled electro-mechanical cases ($e_{ijk} \neq 0$) and decoupled electro-mechanical cases ($e_{ijk} = 0$) are plotted so that the electro-mechanical coupling effects can be clearly observed. Figs. 2(a) and 2(b) are, respectively, the variations of $\Gamma_{2\beta}$ and $\Omega_{2\beta}$ versus the material’s alignment $\gamma$. The results in these two figures correspond to the elements of mechanical deformations induced by mechanical loadings since the sub-indices of $\Gamma_{2\beta}$ and $\Omega_{2\beta}$ all range over $x, \beta = n, s, t$. Note that the curves for $\Gamma_{xx}$ (Fig. 2(a)) with $e_{ijk} = 0$ and $e_{ijk} \neq 0$ overlap since mechanical anti-plane deformations are not affected by the electric loading. For the same reasons, the curves for $\Omega_{xx}$ with $e_{ijk} = 0$ and $e_{ijk} \neq 0$ also overlap as shown in Fig. 2(b). Figs. 3(a) and 3(b) are, respectively, the variations of $\Gamma_{4\beta}$ and $\Omega_{4\beta}$ versus the material’s alignment $\gamma$, but the results are now for the elements of the electrical deformations induced by electrical loading since the sub-indices of $\Gamma_{4\beta}$ and $\Omega_{4\beta}$ all range over $x, \beta = e$ only. The vari-
Fig. 2(a). Variations of $\Gamma_{\alpha \beta}$ ($\alpha, \beta = n, s, v$) which correspond to the mechanical deformations induced by mechanical loadings versus the material’s alignment $\gamma$. Both coupled electro-mechanical cases ($e_{ijkl} \neq 0$) and decoupled electro-mechanical cases ($e_{ijkl} = 0$) are plotted for comparisons.

Fig. 2(b). Variations of $\Omega_{\alpha \beta}$ ($\alpha, \beta = n, s, v$) which correspond to the mechanical deformations induced by mechanical loadings versus the material’s alignment $\gamma$. Both coupled electro-mechanical cases ($e_{ijkl} \neq 0$) and decoupled electro-mechanical cases ($e_{ijkl} = 0$) are plotted for comparisons.

ations of $\Gamma_{x\beta}$ and $\Omega_{x\beta}$ versus $\gamma$ plotted in Figs. 4(a) and 4(b) are the elements of mechanical (electrical) deformations induced by electrical (mechanical) loadings since the sub-index $\alpha = e$ or $\beta = e$ of $\Gamma_{x\beta}$ and $\Omega_{x\beta}$ appears
only once. Note that results for the decoupled electro-mechanical cases presented in Figs. 4(a) and 4(b) are all zeros, as expected, since no coupling occurs when $e_{ijk} = 0$.
To see the whole distribution of the coupled generalized stress components on the boundary induced by the generalized uniform loadings, Figs. 5–7 plot the results for some selected material’s alignments of PZT-4, i.e., for $\gamma = 0^\circ, 30^\circ, 45^\circ, 60^\circ,$ and $90^\circ$. Material PZT-4 is again the transversely isotropic piezoelectric material with

![Diagram](image_url)

**Fig. 4(a).** Variations of $\Gamma_{\alpha\beta}$ ($\alpha, \beta = e$ appears only once, which correspond to the mechanical (electrical) deformations induced by electrical (mechanical) loadings) versus the material’s alignment $\gamma$. Note that the results corresponding to decoupled electro-mechanical cases ($\epsilon_{ik} = 0$) are all identical zeros.

**Fig. 4(b).** Variations of $\Omega_{\alpha\beta}$ ($\alpha, \beta = e$ appears only once, which correspond to the mechanical (electrical) deformations induced by electrical (mechanical) loadings) versus the material’s alignment $\gamma$. Note that the results corresponding to decoupled electro-mechanical cases ($\epsilon_{ik} = 0$) are all identical zeros.

To see the whole distribution of the coupled generalized stress components on the boundary induced by the generalized uniform loadings, Figs. 5–7 plot the results for some selected material’s alignments of PZT-4, i.e., for $\gamma = 0^\circ, 30^\circ, 45^\circ, 60^\circ,$ and $90^\circ$. Material PZT-4 is again the transversely isotropic piezoelectric material with
the poling direction initially parallel to the $x_2$-axis. Note that materials with alignments $\gamma = 30^\circ$, $45^\circ$, and $60^\circ$ belong to the class of monoclinic materials. Figs. 5(a) and 5(b) are, respectively, the distributions of the mechanical stress component $\sigma_{11}(x_1,0)$ and the electric component $D_1(x_1,0)$ on the boundary induced by the mechanical normal loading $\sigma$. Fig. 5(a) shows that the mechanical stress component $\sigma_{11}(x_1,0)$ (normalized by $\sigma$) on the boundary induced by the mechanical normal loading $\sigma$.

![Graph showing distributions of mechanical stress component](image)

**Fig. 5(a).** Distributions of the mechanical stress component $\sigma_{11}(x_1,0)$ (normalized by $\sigma$) on the boundary induced by the mechanical normal loading $\sigma$.

![Graph showing distributions of electric component](image)

**Fig. 5(b).** Distributions of the electric component $D_1(x_1,0)$ (normalized by $\sigma$) on the boundary induced by the mechanical normal loading $\sigma$. 


by $\sigma$ is a constant distribution for $\gamma = 0^\circ$ and $90^\circ$ (since $\Gamma_{\gamma} = 0$ and $\Omega_{\gamma} \neq 0$ for transversely isotropic materials) and $\sigma_{11}(x_1,0)$ is dominated by logarithmic function for all other material alignments (since $\Gamma_{\gamma} \neq 0$ and $\Omega_{\gamma} \neq 0$ for $\gamma \neq 0^\circ,90^\circ$). Under mechanical normal loading, the electric component $D_{1}(x_1,0)$ (normalized by $\sigma$)
Fig. 5(b) has a constant distribution only at $\gamma = 90^\circ$ (since $\Gamma_{en} = 0$ and $\Omega_{en} \neq 0$ for transversely isotropic materials when $\gamma = 90^\circ$). For other material’s alignments, $D_1(x_1,0)$ is dominated by logarithmic function ($\Gamma_{en} \neq 0$ and $\Omega_{en} \neq 0$ for $\gamma \neq 90^\circ$). Figs. 6 and 7 plot the coupled generalized stress components (normalized...
by applied loading) induced by the mechanical shear loading and electric loading, respectively. Both $\sigma_{11}(x_1,0)$ and $D_{11}(x_1,0)$ are dominated by logarithmic function, except for $D_{11}(x_1,0)$ (induced by mechanical shear loading) and $\sigma_{11}(x_1,0)$ (induced by electric loading) which both are constantly distributed at $\gamma = 0^\circ$.

7. Conclusions

The coupled generalized stress components on an anisotropic piezoelectric half-plane boundary under surface electromechanical loading are investigated. Exact expressions for these coupled generalized stress components are derived for general anisotropic piezoelectric materials. The behaviors of the coupled generalized stress components are related to matrices $\mathbf{r}$ and $\mathbf{Q}$, which have the same form as those for the purely elastostatic problem. All elements of both matrices are expressed explicitly in terms of the elastic stiffness for monoclinic piezoelectric materials and transversely isotropic piezoelectric materials. Elements of matrices $\mathbf{r}$ and $\mathbf{Q}$ which correspond to the coupled effects between the mechanical stress components (electric component) induced by electric loadings (mechanical loadings) on the boundary are explored. For isotropic materials, the results for decoupled mechanical deformations and electrical field are both recovered.

Appendix A

Liou and Sung (2007) presented the parameters $Y_{11}$, $Y_{22}$, $Y_{33}$, $Y_{44}$, $Y_{12}$, $Y_{14}$, $Y_{24}$, $\dot{Y}_{12}$, $\dot{Y}_{14}$ and $\dot{Y}_{24}$ in Eq. (4.3) for monoclinic piezoelectric materials with the symmetry plane at $x_3 = 0$. Their results cannot be reduced to the decoupled electro-mechanical case. With arrangements of the eigenvectors that correspond to a new electric deformation in matrix $\mathbf{B}$, i.e., replacing $\mathbf{b}_4 = [-p_4,1,0,-\theta_4]^T$ in Eq. (3.11) of Liou and Sung (2007) by $\mathbf{b}_4 = [-p_4\lambda_4,\lambda_4,0,1]^T$, and then repeating all of the calculation processes described in Liou and Sung (2007), yields new expressions, which are valid even for the decoupled electro-mechanical case. Only expressions different from those given by Liou and Sung (2007) are presented below. The results are

$$Y_{11} = i[a_{11}(1 + \lambda_2\lambda_4) - a_{12}(1 + \lambda_1\lambda_4) + a_{14}(\lambda_1 - \lambda_2)]/|\mathbf{B}|,$$

$$Y_{22} = i[a_{21}(p_2 + \lambda_2\lambda_4p_4) - a_{22}(p_1 + \lambda_1\lambda_4p_4) + a_{24}(\lambda_1p_2 - \lambda_2p_1)]/|\mathbf{B}|,$$

$$Y_{44} = i[a_{41}\lambda_4(p_4 - p_2) + a_{42}\lambda_4(p_1 - p_4) + a_{44}(p_2 - p_1)]/|\mathbf{B}|,$$

$$Y_{12} = -\text{Im}\{[a_{11}(p_2 + \lambda_2\lambda_4p_4) - a_{12}(p_1 + \lambda_1\lambda_4p_4) + a_{14}(\lambda_1p_2 - \lambda_2p_1)]/|\mathbf{B}|\},$$

$$Y_{14} = -\text{Im}\{[a_{11}\lambda_4(p_4 - p_2) + a_{12}\lambda_4(p_1 - p_4) + a_{14}(p_2 - p_1)]/|\mathbf{B}|\},$$

$$Y_{24} = -\text{Im}\{[a_{21}\lambda_4(p_4 - p_2) + a_{22}\lambda_4(p_1 - p_4) + a_{24}(p_2 - p_1)]/|\mathbf{B}|\},$$

$$\dot{Y}_{12} = \text{Re}\{[a_{11}(p_2 + \lambda_2\lambda_4p_4) - a_{12}(p_1 + \lambda_1\lambda_4p_4) + a_{14}(\lambda_1p_2 - \lambda_2p_1)]/|\mathbf{B}|\},$$

$$\dot{Y}_{14} = \text{Re}\{[a_{11}\lambda_4(p_4 - p_2) + a_{12}\lambda_4(p_1 - p_4) + a_{14}(p_2 - p_1)]/|\mathbf{B}|\},$$

$$\dot{Y}_{24} = \text{Re}\{[a_{21}\lambda_4(p_4 - p_2) + a_{22}\lambda_4(p_1 - p_4) + a_{24}(p_2 - p_1)]/|\mathbf{B}|\},$$

where $|\mathbf{B}| = \lambda_4[\lambda_1(p_2 - p_4) + \lambda_2(p_4 - p_1)] + (p_2 - p_1)$, and

$$a_{14}(p_4) = (-\kappa_{11}(p_4) + \kappa_{10}(p_4)\lambda_4(p_4))/\Delta(p_4),$$

$$a_{24}(p_4) = (-\kappa_{21}(p_4) + \kappa_{20}(p_4)\lambda_4(p_4))/\Delta(p_4),$$

$$a_{44}(p_4) = (-\kappa_{41}(p_4) + \kappa_{40}(p_4)\lambda_4(p_4))/\Delta(p_4).$$

Note that expressions for $a_{1k}(p_k)$, $a_{2k}(p_k)$, $a_{4k}(p_k)$ and $\dot{\lambda}_k(p_k)$ ($k = 1, 2$) are the same as those given by Liou and Sung (2007), and
\[
\lambda_4(p_a) = \frac{\sum_{n=0}^{3} \eta_n p_a^n}{\sum_{n=0}^{5} \eta_n p_a^n}, \tag{A.3}
\]

\[
\eta_3 = e_{21} c_{66} e_{22} + c_{12} c_{26} e_{22} - c_{12} c_{26} e_{22} - c_{12} c_{66} e_{22} + c_{16} c_{26} e_{22} - e_{21} c_{26}^2,
\]

\[
\eta_2 = -c_{11} e_{26} e_{22} + e_{26} c_{12}^2 - e_{11} c_{26}^2 - c_{12} e_{22} c_{16} - c_{12} e_{12} c_{66} + c_{16} c_{26} e_{12} + c_{12} c_{26} e_{16} \]

\[
- c_{12} c_{26} e_{21} + c_{22} c_{66} e_{11} + c_{11} c_{26} e_{22} + c_{16} c_{26} e_{21} - c_{12} c_{26} e_{16},
\]

\[
\eta_1 = c_{11} c_{26} e_{12} + c_{12} c_{16} e_{26} + c_{16} c_{26} e_{11} - c_{12} c_{16} e_{12} + c_{11} c_{26} e_{26} - e_{22} c_{16}^2 \]

\[
+ c_{11} c_{66} e_{22} - c_{12} c_{66} e_{21} + c_{16} c_{26} e_{21} - c_{12} c_{66} e_{11} + e_{16} c_{12}^2,
\]

\[
\eta_0 = -c_{11} c_{26} e_{16} + c_{12} c_{66} e_{12} + c_{16} c_{26} e_{11} - c_{12} c_{66} e_{11} + c_{12} c_{16} e_{16} - e_{12} c_{16}^2.
\]

\[
\zeta_5 = -c_{22} e_{26}^2 - c_{66} e_{26}^2 + x_{22} c_{26}^2 - c_{22} c_{66} x_{22} + 2 c_{26} e_{22} e_{26},
\]

\[
\zeta_4 = 2 c_{12} c_{26} x_{22} - c_{22} c_{16} e_{26} - c_{66} e_{12} e_{22} + x_{12} c_{26}^2 + 2 c_{12} e_{22} e_{26} + c_{66} e_{12} e_{26} \]

\[
- 2 c_{16} e_{26}^2 + 2 c_{26} e_{22} e_{26} + 2 c_{16} e_{22} x_{22} - c_{66} e_{22} x_{12} - 2 c_{22} e_{26} e_{26},
\]

\[
\zeta_3 = 2 c_{66} e_{22} e_{21} + c_{12} c_{12} e_{26} - c_{22} e_{21} e_{16} - c_{11} e_{22}^2 - 2 c_{12} e_{26}^2 - 2 c_{16} e_{12} e_{22} - 2 c_{22} e_{26}^2 \]

\[
+ 2 c_{12} e_{22} e_{21} - c_{22} e_{21} e_{16} + 2 c_{12} c_{26} x_{12} + x_{22} e_{12}^2 + 2 c_{12} c_{66} x_{22} - 2 c_{16} c_{26} x_{22} \]

\[
+ c_{12} c_{26} e_{11} + c_{26} e_{22} e_{11} - c_{11} c_{22} x_{12} - 2 c_{26} e_{21} e_{26} - 2 c_{16} c_{22} x_{12} + 2 c_{66} e_{12} e_{21} - 2 c_{16} c_{22} e_{26},
\]

\[
\zeta_2 = -2 c_{26} e_{26}^2 - 2 c_{16} e_{26} x_{12} - c_{16} e_{12} e_{26} + 2 c_{12} c_{66} x_{12} + c_{12} c_{12} x_{21} + 2 c_{12} e_{16} e_{26} \]

\[
+ 2 c_{12} c_{16} x_{22} - c_{16} e_{22} e_{16} - c_{26} e_{16} e_{21} - c_{11} c_{22} x_{12} + c_{66} c_{12} e_{21} - c_{22} e_{11} e_{21} \]

\[
+ 2 c_{12} e_{21} e_{26} - 2 c_{11} e_{22} e_{26} - c_{66} e_{11} e_{26} + c_{16} e_{11} e_{22} - 2 c_{11} c_{26} x_{22}
\]

\[
+ c_{12} e_{11} e_{22} + x_{12} e_{12}^2 - c_{11} e_{12} e_{22},
\]

\[
\zeta_1 = -c_{11} c_{26} x_{12} + c_{12} c_{16} e_{21} + 2 c_{11} c_{26} x_{12} + c_{12} c_{12} e_{21} + x_{22} e_{12}^2 - c_{11} e_{12}^2 - c_{11} e_{12} e_{22} \]

\[
+ 2 c_{12} c_{12} x_{12} + c_{12} e_{12} e_{21} - c_{66} e_{21}^2 + 2 c_{12} e_{21} e_{26} + c_{16} e_{11} e_{22} - 2 c_{26} e_{11} e_{21} - c_{11} e_{12} e_{26},
\]

\[
\zeta_0 = -c_{11} c_{26} x_{12} - c_{66} e_{11} e_{21} + x_{12} e_{12}^2 + c_{16} e_{12} e_{21} + c_{16} e_{11} e_{26} - c_{11} e_{12} e_{26}.
\]

### Appendix B

The material constants for practical piezoelectric ceramic PZT-4 (Ou and Chen, 2003) are

\[
[c_{\alpha\beta}] = \begin{bmatrix}
139.0 & 74.3 & 77.8 & 0 & 0 & 0 \\
74.3 & 113.0 & 74.3 & 0 & 0 & 0 \\
77.8 & 74.3 & 139.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 25.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 30.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 25.6
\end{bmatrix} \quad (10^9 \text{ N m}^{-2}),
\]

\[
[e_{\alpha\alpha}] = \begin{bmatrix}
-6.98 & 13.8 & -6.98 & 0 & 0 & 0 \\
0 & 0 & 0 & 13.4 & 0 & 0 \\
6.00 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5.47 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.00 & 0
\end{bmatrix} \quad (\text{C m}^{-2}),
\]

\[
[z_{i\alpha}] = \begin{bmatrix}
6.00 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.47 & 0 & 0 & 0 & 0 \\
0 & 0 & 6.00 & 0 & 0 & 0
\end{bmatrix} \quad (10^{-9} \text{ C V}^{-1} \text{ m}^{-1}),
\]

and the eigenvalues corresponding to this practical piezoelectric ceramic are

\[
p_1 = 0.2733 + 1.0872i, \quad p_2 = -0.2733 + 1.0872i, \quad p_3 = 1.0933i, \quad p_4 = 1.1902i.
\]

The elements of matrices \( L^{-1} \) and \( SL^{-1} \) for this practical piezoelectric ceramic are
$L^{-1} = \begin{bmatrix}
Y_{11} & 0 & 0 & 0 \\
0 & Y_{22} & 0 & Y_{24} \\
0 & 0 & Y_{33} & 0 \\
0 & Y_{24} & 0 & Y_{44}
\end{bmatrix} = \begin{bmatrix}
0.01763 & 0 & 0 & 0 \\
0 & 0.01752 & 0 & 0.02215 \\
0 & 0 & 0.03573 & 0 \\
0 & 0.02215 & 0 & -0.08765
\end{bmatrix},$

$SL^{-1} = \begin{bmatrix}
0 & -\tilde{Y}_{12} & 0 & -\tilde{Y}_{14} \\
\tilde{Y}_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\tilde{Y}_{14} & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & -0.007612 & 0 & 0.01881 \\
0.007612 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-0.01881 & 0 & 0 & 0
\end{bmatrix}.$

References


