Ordering Policy For Spare Preventive Replacement

Jih-An Chen∗

Department Department of Business Administration
Kao-Yuan University
Kaohsiung, Taiwan

ABSTRACT

We develop a preventive replacement policy of spare ordering under cost effectiveness criterion. The spare unit for replacement is available only by order and the lead time for delivery the spare due to regular or expedited ordering follows general distributions. Introducing costs due to ordering, repairs, downtime and replacements, as well as the salvage value of system, we derive the expected cost effectiveness per unit time in the long run as a criterion of optimality. There exists a finite and unique optimum policy of ordering time which maximizes the expected cost effectiveness.

1. INTRODUCTION

One important area of interest in reliability theory is the study of various maintenance policies in order to reduce the operating cost and the risk of a catastrophic breakdown. Many preventive maintenance policies have been proposed and discussed in the past four decades. See, Barlow and Hunter (1960), Nakagawa and Kowada (1983), Sheu et al. (1994), Wang (2002), Chien (2005), Chen et al. (2007) for example. However, this might not be true on some occasions. For instance, it is usual in commercial industries that only one spare unit, which can be delivered by order, is available for replacement. That is to say, introducing the random lead time is essential and practical. Once we take account of the random lead time, we should consider an ordering policy that determines when to order a spare and when to replace the operating unit after it has begun operating. Preceding research of ordering policies have assumed the costs for preventive and corrective replacements are equal (for example, see Chien (2005), Dohi et al. (1998), Sheu and Liou (1994)), which imply no particular need for preventive replacement.

Moreover, most of them seek the optimum ordering policy by minimizing the expected cost per unit time in the long run as a criterion of optimality. In general, however, the policy maximizing profit rate does not discriminate among large and small investments. Thus, the policy mentioned above might have this property. Therefore, the cost effectiveness as an alternative criterion is suitable for reflecting efficiency per dollar spent. The cost effectiveness is defined as (s-availability)/(s-expected cost rate) which reflects the efficiency per dollar outlay. Chien and Chen (2007) also determined the optimal ordering time based on the cost effectiveness. This criterion is useful for the effective use of available money. Especially, this criterion is useful when the benefits obtained from investment are difficult to quantify. As an example, national security may benefit from weapon systems, but expressing the gains in financial terms is quite challenging.

In this paper, we present a spare ordering policy for preventive replacement. And, a general repairable system is also considered. In the general repair model, when the system fails at age $t$, type I failure occurs with probability $q(t)=1-p(t)$ and type II failure occurs with probability $p(t)$, $0\leq p(t)\leq 1$ and $p(t)$ is non-decreasing in age $t$. Type I failure is assumed to be minor, and can thus be rectified through minimal repair; while type II failure is catastrophic, and can be removed only by replacement. Minimal repair means that the repaired system is returned in the same condition as it was, that is to say, the failure rate of the repaired system remains the same as it was just prior.

∗ Corresponding author: Tel.: (886) 7-607-7074; Fax: (886) 7-607-7116; E-mail: jachen@cc.kyu.edu.tw
to failure. The repair cost depends on the age and the number of minimal repair, and the cost for corrective replacement is larger than that for preventive replacement. The cost effectiveness for operating the system in an infinite time-span is adopted as a criterion of optimality.

2. MODEL FRAMEWORK

We consider the following operating scheme:

1. The original one-unit system begins operating at time 0 and the operation system has two types of failures which could occur at any age \( t \). A type I failure (minor failure) occurs with probability \( q(t) \) and is corrected with minimal repair. A type II failure (major failure) occurs with probability \( p(t) = 1 - q(t) \) and is followed by unit replacement. Replacements are made perfectly and do not affect the system’s characteristics.

2. All failures are instantly detected and is repaired instantaneously if it is a type I failure. If the type II failure occurs before a pre-specified ordering time \( T \), then the system is shut down and an expedited order is made immediately at the failure time instant. Otherwise, the regular order is made at time \( T \).

3. When the expedited order is made, then the failed system is replaced by the expedited ordered spare as soon as the spare is delivered. The lead time for delivery that spare is a generally distributed random variable with density function \( W(t) \) and finite mean \( \mu_w \). And, when the regular order is made, then the original system, no matter operating or not, is replaced by the regular ordered spare as soon as the spare is delivered. The lead time for delivery that spare is a generally distributed random variable with density function \( Z(t) \), distribution function \( Z(t) \) and finite mean \( \mu_z \) (\( \mu_z \geq 0 \)).

4. After a replacement, the procedure is repeated.

We also define the following various costs for operating this ordering-replacement procedure:

1. The cost for an expedited order is \( c_e \), and the cost for a regular order is \( c_r \) (\( c_e \geq c_r > 0 \)).

2. The cost for a corrective replacement is \( c_c \), and the cost for a preventive replacement is \( c_p \) (\( c_c \geq c_p > 0 \)).

3. The cost rate resulting from the system down is \( c_d \).

4. The salvage value per unit time for the residual lifetime of an un-failed system is \( v_s \).

5. The cost of the \( i \)-th minimal repair at age \( t \) is \( \phi(K(t), k_i(t)) \) where \( K(t) \) is the age-dependent random part, \( k_i(t) \) is the deterministic part that depends on the age and the number of minimal repair, and \( \phi \) is a positive non-decreasing continuous function of age \( t \).

3. COST FORMULATION

According to the replacement scheme described in Section 2, the following three mutually exclusive and exhaustive states (possibilities) between successive replacements can be defined as follows:

(i). State 1: A type II failure occurs before the scheduled ordering time \( T \), i.e., an expedited order is made and the failed system is replaced correctly by the spare as soon as the spare is delivered.

(ii). State 2: A type II failure occurs between \( T \) and the arrival of the ordered spare; i.e., a regular order is made and the failed system is replaced correctly by the spare as soon as the spare is delivered.

(iii). State 3: No type II failure occurs before the arrival of the ordered spare; i.e., a regular order is made and the un-failed system is replaced preventively by the spare as soon as the spare is delivered.

In order to develop the cost model of spare ordering, the following lemma from Sheu and Liou (1994) is required and stated as follow.

Lemma 1. Let \( \{M(t), t \geq 0\} \) be a non-homogeneous Poison process with intensity function \( q(t)\lambda(t) \).
\[ E[M(t)] = \int q(x)r(x)dx. \]

Assume that the successive arrival time at time \( S_i (i = 1, 2, \ldots) \) will be incurred a cost of \( \phi(K(S_i), k(S_i)) \). Suppose that \( K(t) \) at age \( t \) is a random variable with finite mean \( E[K(t)] \). If \( \pi(t) \) is the total cost incurred over \([0, t)\), then

\[ E[\pi(t)] = E \left[ \sum_{i=1}^{M(M)} \phi(K(S_i), k(S_i)) \right] = \int_0^T \theta(x)q(x)r(x)dx, \text{ where } \theta(t) = E_{M(\cdot)}[\phi(K(t), k_{M(\cdot)}(t))]. \]

According to the above three mutually exclusive and exhaustive states, the expected cycle length is

\[ \int_0^T \int_0^T (x+y)dF_p(x)dZ(y) + \int_0^T \int_y^{T+y} (T+y)dF_p(x)dW(y) + \int_0^T \int_0^{T+y} (T+y-x)dF_p(x)dW(y). \]

In view of the fact that downtime only occurs in the states 1 and 2, thus the expected downtime per cycle \( D(T) \) can be expressed as

\[ \int_0^T \int_0^T dF_p(x)dZ(y) + \int_0^T \int_y^{T+y} (T+y-x)dF_p(x)dW(y). \]

And, the uptime per cycle is the cycle length minus the downtime per cycle, the expected uptime per cycle \( U(T) \) is given by

\[ \mu_r + \int_0^T \bar{F}_p(x)dx - \int_0^T \int_y^{T+y} F_p(x)dW(y). \]

The expected cost per cycle can be expressed as the sum of the cost for ordering, minimal repair, downtime and replacement, and the salvage value of the system. It is obviously that the expected ordering cost per cycle is given by

\[ c_o \cdot F_o(T) + c_r \cdot \bar{F}_p(T) = c_r + (c_o - c_r)F_p(T). \]

And, the expected downtime cost per cycle is

\[ c_d \cdot D(T) = c_d \left\{ \int_0^T \int_y^{T+y} F_p(x)dW(y) - (\mu_r - \mu_r)F_p(T) \right\}. \]

For the cost due to replacement, since the corrective replacement occurs in the states 1 and 2, and the preventive replacement cost occurs in the state 3. Thus the expected replacement cost per cycle is given by

\[ c_e \left\{ F_p(T) + \int_0^T \int_y^{T+y} dF_p(x)dW(y) \right\} + c_p \cdot \int_0^T \int_y^{T+y} dF_p(x)dW(y). \]

And, for the cost due to minimal repair, there are three mutually exclusive and exhaustive possibilities for the minimal repair costs that exist in every cycle, which can be expressed as:

\[
\begin{aligned}
&\sum_{i=1}^{M(X_1)} \phi(K(S_i), k(S_i)), \text{ if } X_1 < T, \\
&\sum_{i=1}^{M(X_2)} \phi(K(S_i), k(S_i)), \text{ if } T \leq X_1 < T + Y_r, \\
&\sum_{i=1}^{M(T+Y_r)} \phi(K(S_i), k(S_i)), \text{ if } X_1 \geq T + Y_r,
\end{aligned}
\]

where \( X_1 \) is the time to first type II failure, and \( Y_r \) is the random lead time for delivery the regular ordered spare.

Thus, take expectation and by using Lemma 1, the expected minimal repair cost per cycle is

\[
\begin{aligned}
&\int_0^T E \left\{ \sum_{i=1}^{M(X_1)} \phi(K(S_i), k(S_i)) \right\}dF_p(x) + \int_0^T \int_y^{T+y} E \left\{ \sum_{i=1}^{M(X_2)} \phi(K(S_i), k(S_i)) \right\}dF_p(x)dW(y) \\
&+ \int_0^T \bar{F}_p(T+y)E \left\{ \sum_{i=1}^{M(T+Y_r)} \phi(K(S_i), k(S_i)) \right\}dW(y)
\end{aligned}
\]
It seems reasonable that salvage value of a used unit, which is still operable, is proportional to the expected residual lifetime. Salvage value occurs only in the state 3, and the expected salvage value per cycle is given by

\[
v_s \cdot \int_0^\infty \int_{y_T}^\infty (x - T - y)dF_p(x)dW(y) = v_s \cdot \int_0^\infty \int_{y_T}^\infty \bar{F}_p(x)dx dW(y).
\]

Thus, the expected operational cost per cycle can be expressed as

\[
C(T) = (c_s + c_p) + \left[ (c_e - c_r) - c_d(\mu_e - \mu_s) \right]F_p(T) + \int_0^{T+y} \bar{F}_p(x)\theta(x)q(x)r(x)dx dW(y) + (c_e - c_p)\int_0^\infty F_p(T + y)dy + c_d \cdot \int_0^T \int_{y_T}^\infty F_p(x)dx dW(y) - v_s \cdot \int_0^\infty \int_{y_T}^\infty \bar{F}_p(x)dx dW(y).
\]

Given that each replacement is a regeneration point, the cost effectiveness defined as \((s\text{-availability})/(s\text{-expected cost rate})\) can be rewritten as \((\text{expected uptime in a cycle})/(\text{expected cost per cycle})\). Consequently, the cost effectiveness can be expressed as

\[
CE(T) = \frac{U(T)}{C(T)}.
\]

4.OPTIMIZATION ANALYSIS

Define the both functions:

\[
r_{p,r}(t) = f_p(t) / \bar{F}_p(t + y) \quad \text{and} \quad F_{p,r}(t) = [F_p(t + y) - F_p(t)] / \bar{F}_p(t + y).
\]

The following lemma is required and supportive to examine the existence and uniqueness of the optimum ordering policy.

Lemma 2. Both \(r_{p,r}(t)\) and \(F_{p,r}(t)\) are strictly increasing if \(r(t)\) is strictly increasing and \(p(t)\) is non-decreasing.

Define the numerator of the derivative of \(CE(T)\) divided by \(\bar{F}_p(T + y)\) as \(\zeta(T)\):

\[
\zeta(T) = C(T) - U(T) \times \left[ (c_e - c_r) - c_d(\mu_e - \mu_s) \right]F_{p,r}(T) + (\theta(T + y)q(T + y) + (c_e - c_p)p(T + y))r(T + y) + c_d F_{p,r}(T) + v_s \Bigg]
\]

Then, the main results concerning the optimal spare ordering time \(T^*\) which maximizes cost effectiveness are summarized below.

Theorem 1. Supposing \([\theta(t)q(t) + (c_e - c_p)p(t)]r(t)\) is continuous and strictly increasing in age \(t\), and the condition \((c_e - c_r) \geq c_d(\mu_e - \mu_s)\) holds, then (1). If \(\zeta(0) \leq 0\), then the optimum spare ordering time \(T^* = 0\), i.e., a regular order is made at the same instant when a new system is put in service and never place an expedited order. (2). If \(\zeta(0) > 0\) and \(\zeta(\infty) < 0\), then there exists a finite and unique optimum spare ordering time \(T^*\) satisfying \(\zeta(T^*) = 0\).
(3) If $\zeta(\infty) \geq 0$, then the optimum spare ordering time $T^* \to \infty$, i.e., an expedited order is made at the instant of failure and never place a regular order.

5. CONCLUSION

Under the above cost model to study the preventive replacement policy of spare ordering, we derived the existence and uniqueness of the optimum ordering policy under cost effectiveness criterion. It is worth pointing out that, if the spare is inexpensive, quantity purchases might be practical, and it is natural to consider a stocking policy to determine how many spares to purchase with each order. Indeed, the spare ordering in this paper are usually components in a high-cost complex system. The sufficient condition in the above theorem, $(c_r - c_e) \geq c_r (\mu_e - \mu_r)$, for the optimal spare ordering policy has been far and wide used in spare ordering policies, for example, see Chien (2005). However, it should be noted that the assumption does not economically justify placing an expedited order since the additional cost for the expedition $(c_r - c_e)$ is larger than the savings obtained from the expedition $c_e (\mu_e - \mu_r)$. Hence it is meaningful only when there exists such intangibles as loss of goodwill, reputation and credit which are difficult to be quantified and included in downtime cost.

REFERENCES