Optimal Age-Replacement Model with Minimal Repair Based on Cumulative Repair Cost Limit and Random Lead Time

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Abstract - In this study, we consider an age-replacement model with minimal repair based on a cumulative repair cost limit and random lead time for replacement delivery. A cumulative repair cost limit policy uses information about a system's entire repair cost history to decide whether the system is repaired or replaced; a random lead time models delay in delivery of a replacement once it is ordered. A general cost model is developed for the average cost per unit time based on the stochastic behavior of the assumed system, reflecting the costs of both storing a spare and of system downtime. The minimum-cost policy time is derived, its existence and uniqueness is shown, and structural properties are presented.

Keywords - Minimal repair, Replacement, Repair cost limit, Inventory, Maintenance.

I. INTRODUCTION

The main deficiency of the traditional repair cost limit policy is that the decision to repair or replace a system depends only on the cost of a single repair. Under this type of policy, a system with frequent failures and consequently high accumulated repair costs will continue to be repaired rather than replaced provided that the cost of each repair is less than the pre-determined limit. It is clearly more reasonable to base the decision to repair or replace a system on its entire repair history, thereby avoiding short-sighted management of failure-prone equipment. Natural improvements along these lines, e.g., incorporating repair cost rates per unit time or accumulated repair costs in the decision to repair or replace a system, have been proposed. Beichelt (1982) proposed a replacement policy based on limits for the repair cost rate: the system is replaced as soon as the repair cost rate exceeds a given level. Beichelt (2001) further proposed a cumulative repair cost limit replacement policy: the system is replaced as soon as the accumulated maintenance cost $C(t)$ reaches or exceeds a given cutoff. However, $C(t)$ is given exogenously and does not exploit the lifetime repair history.

Moreover, the replacement policies considered to date (including those mentioned above) all assume that at any time there is an unlimited supply of items available for immediate replacement. This assumption might be quite unrealistic; e.g., it is commonplace in commercial industries that only one spare unit is available for replacement. In this case, a random lead time for spare unit delivery should not be neglected. It is essential to consider and analyze the effect of the random lead time. Nakagawa and Osaki (1974), Osaki and Yamada (1976), Sheu and Griffith (2001), and Sheu and Chien (2004) all considered the random lead time for replacement when they discussed the optimum preventive replacement policies.

In this paper, a generalized age-replacement policy based on a cumulative repair cost limit is presented, and the random lead time for replacement is also considered. A model is developed for the average cost per unit time based on the stochastic behavior of the assumed system. The model reflects the cost of storing a spare as well as the cost of system downtime. The minimum-cost policy time is derived and discussed. Special cases are addressed.

II. THE MODEL

In the age-replacement policy, planned (scheduled) replacements occur whenever an operating system reaches age $T$ and a spare unit is available for replacement. We say that a replacement cycle is exchanged if a replacement occurs, where a replacement cycle is the time interval between the installation of the system and the first replacement or between consecutive replacements. In this framework, replacement cycles constitute a regenerative process.

A. Preliminaries

Assume that the system has a failure time distribution $F(t)$ with finite mean $\mu$ and a density function $f(t)$. Then the failure rate (or hazard rate) is $r(t)=f(t)/F(t)$ and the cumulative hazard $\Lambda(t)=\int_0^t r(u)du = -\ln(F(t))$ is the mean number of failures that occur in $[0, t]$, where $F(t)=1-F(t)$.

System failures can be divided into two categories: a type I failure is a minor failure that can be corrected by minimal repair and a type II failure is a catastrophic failure in which the system is damaged beyond repair. We assume a type I failure occurs with probability $1-p$ and a type II failure occurs with probability $p$ $(0<\rho<1)$. We further assume that the failure rate $r(t)$ is continuous, monotone increasing, and remains undisturbed by minimal repair.

Let $Y$ be the time to the first type II failure. From Eq. (12) of Beichelt (1993), the survival function of $Y$ is directly obtained:
\[ F_Y(t) = P(Y > t) = \exp(-p\Lambda(t)) \]  

On the other hand, let \( N(t) \) be the random number of minimal repairs in \((0, \min(Y, t))\). Then, the probability of \( k \) type I failures (minimal repairs) in \([0, t] \) should be defined and written as (from page 58 of Beichelt (1993)): 

\[
p_i(t) = P(N(t) = k | Y = i) = P(N(t) = k | Y > t)
\]

\[
= \left[ (1 - p)\Lambda(t) \right]^k \exp(- (1 - p)\Lambda(t)).
\]  

Moreover, let \( S_j \) \((S_j = 1, 2, 3, \ldots)\) denote the occurrence time of the \( j \)-th type I system failure, where \( S_0 = 0 \); then the conditional distribution function of the random variable \( S_j \) is given by 

\[ F_{S_j}(t) = P(S_j \leq t | Y > t) = P(N(t) \geq j | Y > t) \] 

\[
= \sum_{j \geq 1} p_i(t),
\]

and 

\[ F_{S_j}(t) = \sum_{j=0}^{\infty} p_i(t). \]  

We assume that the repair costs \( W_i \) for the \( i \)-th type I failure are non-negative i.i.d. random variables with a probability distribution function \( G(x) = P(W_i \leq x) \) with a finite mean \( c_i \) \((i=1, 2, 3, \ldots)\). Let \( Z_j = \sum_{i=1}^{\infty} W_i \) be the accumulated repair cost until the \( j \)-th type I failure; then the distribution function of \( Z_j \) is given by 

\[ P(Z_j \leq y) = G^{(j)}(y) = \left\{ \begin{array}{ll}
G * G * \cdots * G(y), & j = 1, 2, 3, \ldots \\
0, & j = 0.
\end{array} \right. \]

which is the \( n \)-fold Stieltjes convolution of the distribution \( G(x) \) of itself. 

B. Replacements

In this general replacement model with random lead time, replacements and repairs are executed according to the following scheme. 

1. A planned preventive replacement is carried out at age \( T \) at a cost \( c_p \). 
2. An unplanned replacement due to a type I failure is executed at the time of failure at a cost \( c_p \) when the accumulated repair cost till this minor failure exceeds a pre-determined limit \( \xi \) (i.e., it can be regard as a type I failure replacement). 
3. An unplanned replacement due to a type II failure occurs at the time of failure at a cost \( c_p \) (i.e., it can be regard as a type II failure replacement). However, if the accumulated repair cost up to and including a type I failure is less than \( \xi \), then minimal repair occurs instead of replacement. 
4. If an ordered spare unit has not arrived when a replacement is necessary, the replacement execution must be postponed. In other words, if a spare unit is available, the system is replaced preventively at age \( T \); or at the \( j \)-th type I failure at which the accumulated repair cost \( Z_j \) exceeds the pre-determined limit \( \xi \), or at the first type II failure, whichever occurs first. Once ordered, the spare unit delivery time (i.e., the lead time) \( X_\zeta \) has distribution function \( L(t) \), density function \( l(t) \), survival function \( \bar{L}(t) \), and finite mean \( \text{E}(X_\zeta) = \mu_\zeta \). 

5. We assume that the operation of the original system and the random lead time both begin at time 0; that is, a spare unit is ordered immediately upon system installation. If the spare unit arrives before system replacement, then the replacement can be made immediately when needed; otherwise, the system remains down until the spare unit is delivered. 

To be more precise, we define the following six mutually exclusive and exhaustive states between successive replacements. The corresponding probabilities of these six states are also derived.

- **State 1.** The system has a critical type I failure and is replaced.

Here 'a critical type I failure' means 'a type I failure which causes the excess of the accumulated maintenance cost over \( \xi \) at the first time'. This case occurs when a critical type I failure precedes time \( T \), no type II failure has occurred, the accumulated repair cost exceeds the pre-determined limit \( \xi \) for the first time, and an ordered spare has already arrived and is waiting in storage. The probability of this replacement state is given by 

\[
\sum_{j=0}^{\infty} \int_0^{\mu_\zeta} \int_0^{\xi} P(Z_j < \xi < Z_j) L(t) F_j(t) dF_{S_j}(t)
\]

\[
= \sum_{j=0}^{\infty} \int_0^{\mu_\zeta} \left[ G^{(j)}(\xi) - G^{(j-1)}(\xi) \right] L(t) F_j(t) (1 - p(r)) p_j(t) dt \]  

- **State 2.** The system has a critical type I failure and awaits replacement. 

This case occurs when a critical type I failure is occurred, no type II failure has occurred, the accumulated repair cost exceeds the pre-determined limit \( \xi \) for the first time, but an ordered spare unit has not yet arrived. The probability of this replacement state is given by 

\[
\sum_{j=0}^{\infty} \int_0^{\infty} \int_0^{\xi} P(Z_j < \xi < Z_j) L(t) F_j(t) dF_{S_j}(t)
\]

\[
= \sum_{j=0}^{\infty} \left[ G^{(j)}(\xi) - G^{(j-1)}(\xi) \right] L(t) F_j(t) (1 - p(r)) p_j(t) dt \]  

- **State 3.** The system has a type II failure and is replaced. 

This case occurs when a type II failure precedes time \( T \), the accumulated repair cost is less than \( \xi \), and an ordered spare is waiting in storage. The probability of this replacement state is given by 

\[
\sum_{j=0}^{\infty} \int_0^{\infty} L(t) P(N_j(t) = j | Y = t) P(Z_j < \xi) dF_j(t)
\]

\[
= \sum_{j=0}^{\infty} \left[ G^{(j)}(\xi) \right] L(t) F_j(t) p_j(t) dt \]  

- **State 4.** The system has a type II failure and awaits replacement. 

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This case occurs when a type II failure is occurred, the accumulated repair cost is less than $\xi$, but an ordered spare has not yet arrived. The probability of this replacement state is given by

$$\sum_{j=0}^{\infty} \tilde{L}(t) p_j(t)p(Z_j < \xi) d\tilde{F}_j(t)$$

$$= \sum_{j=0}^{\infty} G^{(j)}(\xi) I(L(t)\tilde{F}_j(t) pr(t)p_j(t)dt).$$

(8)

This is another kind of delayed corrective replacement.

- **State 5.** The system is replaced preventively at time $T$.

This case occurs when the predetermined age-replacement time $T$ precedes a type II failure, the accumulated repair cost is less than $\xi$, and an ordered spare is waiting in storage. The probability of this replacement state is given by

$$\sum_{j=0}^{\infty} P(X_j \leq T)P(Y_j > T)P(N_j(T) = j \mid Y_j > T)p(Z_j < \xi)$$

$$= L(T)\tilde{F}_j(T)\sum_{j=0}^{\infty} G^{(j)}(\xi) p_j(T).$$

(9)

- **State 6.** The system is replaced preventively after time $T$.

This case occurs when the predetermined age-replacement time $T$ precedes a type II failure, the accumulated repair cost is less than $\xi$, but an ordered spare has not yet arrived. In this case, the system is preventively replaced when the spare unit arrives. The probability of this replacement state is given by

$$\sum_{j=0}^{\infty} \tilde{F}_j(t)P(N_j(t) = j \mid Y_j > t)p(Z_j < \xi)dL(t)$$

$$= \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)p_j(t)dt.$$ 

(10)

This is a delayed preventive replacement.

Moreover, we also need the following assumptions.

1. The system is monitored continuously and failures are detected immediately.
2. Repairs and replacements are completed instantaneously.
3. Replacements are perfect and do not affect the system characteristics.
4. Costs for replacement are ordered $c_0 < c_1 < c_2$. $c_1 > c_0$ means that the corrective replacement cost is greater than the preventive replacement cost; $c_2 > c_1$ means that a catastrophic failure is more costly than a minor failure. In other words, $c_2$ includes additional system restoration costs.
5. The cost per unit time to store a spare unit is $c_b$.
6. The cost per unit time for a down system is $c_d$.

**C. Formulation**

For our model, let $U_i$ denote the length of the $i$-th successive replacement cycle for $i = 1, 2, 3, \ldots$; and let $V_i$ denote the operational cost over the renewal interval $U_i$.

Thus $\{U_i, V_i\}$ constitutes a renewal reward process. If $D(t)$ denotes the expected cost of operating the system over time interval $[0, t]$, then it is well-known that

$$\lim_{t \to \infty} \frac{D(t)}{t} = \frac{E(D_i)}{E(U_i)},$$

(11)

(see, e.g., Ross (1970), page 52). We shall denote the right-hand side of (12) by $C(T)$.

Let $Y_1, Y_2, \ldots$ be independent copies of $Y$. According to the replacement scheme that was described in the subsection B, the length of first replacement cycle $U_1$ can be expressed as

$$U_1 = \begin{cases} S_i, & \text{if state 1}, \\ X_1, & \text{if state 2}, \\ Y_1, & \text{if state 3}, \\ X_1, & \text{if state 4}, \\ T, & \text{if state 5}, \\ X_1, & \text{if state 6}. \\ \end{cases}$$

(12)

and the expected length of a replacement cycle is

$$E[U_i] = \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)p_j(t)dt + \mu_i.$$ 

(13)

Similarly, the operating cost over the first replacement cycle can be expressed as:

$$V_i = \begin{cases} c_1 + \sum_{n=1}^{\infty} c_2 (S_i - X_i), & \text{if state 1}, \\ c_1 + \sum_{n=1}^{\infty} c_1 (X_i - S_i), & \text{if state 2}, \\ c_2 + \sum_{n=1}^{\infty} c_1 (Y_i - X_i), & \text{if state 3}, \\ c_2 + \sum_{n=1}^{\infty} c_1 (Y_i - X_i), & \text{if state 4}, \\ c_2 + \sum_{n=1}^{\infty} c_1 (T - X_i), & \text{if state 5}, \\ c_d + \sum_{n=1}^{\infty} c_1 (T - X_i), & \text{if state 6}. \\ \end{cases}$$

(14)

so the expected operating cost in a replacement cycle is given by

$$E[V_i] = c_1 - (c_1 - c_2) \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)p_j(t)dt$$

$$- (c_1 - c_2) L(T)\tilde{F}_j(T) \sum_{j=0}^{\infty} G^{(j)}(\xi) p_j(T)$$

$$+ (c_1 - c_2) \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)(1 - p(t))d\tilde{F}_j(t)$$

$$+ c_d \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)(1 - p(t))d\tilde{F}_j(t)$$

$$+ c_0 \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)p_j(t)dt$$

$$+ (c_1 - c_2) \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)(1 - p(t))d\tilde{F}_j(t)$$

$$+ c_d \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)(1 - p(t))d\tilde{F}_j(t)$$

$$+ c_1 \left\{ \mu_c - \sum_{j=0}^{\infty} G^{(j)}(\xi) I(t)\tilde{F}_j(t)p_j(t)dt \right\}. $$

(15)
Therefore, the long-run expected cost per unit time for operating the system is given by
\[
C(T) = \frac{E[V]}{E[U]},
\]
(16)

**D. Optimization**

A necessary condition that a finite \( T^* \) minimizes \( C(T) \) can be obtained by differentiating \( C(T) \) with respect to \( T \) and setting it equal to zero. Then, the structural properties of the optimal \( T^* \) that minimize \( C(T) \) can be obtained.

**III. SPECIAL CASES**

If the repair cost limit \( \xi = \infty \); that is, all type I failures can be rectified through minimal repairs, then the replacement model will reduce to Sheu and Griffith (1996). Put \( \xi = \infty \) in equations (13), (15) and (16), then the cost rate function will become
\[
C(T) = \frac{\xi}{\int_0^T L(t)\bar{F}_r(t)dt + \mu_*},
\]
(17)
where
\[
\xi = \left[ L(T)\bar{F}_r(T) + \int_0^T \bar{F}_r(t) dL(t) \right]
+ \left[ \bar{L}(t) + \int_0^T \bar{F}_r(t) dL(t) \right]
+ \left[ \int_0^T \bar{F}_r(t)(1-p) r(t) dt + \int_0^T \bar{L}(t)\bar{F}_r(t)(1-p) r(t) dt \right]
+ \int_0^T L(t)\bar{F}_r(t)dt + \int_0^T \bar{L}(t)F_r(t)dt.
\]

In Sheu and Griffith (1996), they considered that the occurrence probability of a type II failure and minimal repair cost are all depends on age \( t \). However, if we let the type II failure probability is \( p(t)=p \) and minimal repair cost is \( h(t)=c_* \) in equation (8) of Sheu and Griffith (1996), then it will agree with (17).

**IV. CONCLUDING REMARKS**

In this paper, an age-replacement model with minimal repair is presented, which is based on a cumulative repair cost limit and random lead time for replacement. The long-run expected cost per unit time for operating the system were developed which incorporating costs due to holding a spare unit, shortage, minimal repairs, and different kinds of replacement state. The optimal preventative replacement age which minimizes that cost rate function is derived, its existence and uniqueness can be shown based on the cost rate function, and then the structural properties can be obtained. One special case is dealt with. Because the model and its analysis are general, several existing results are shown to be subsumed by this model.

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