Abstract: This paper presents a robust gain-scheduled control approach for planar vertical take-off aircraft dynamics. The uncertain properties of the aircraft mass, moment of inertia, and the non-minimum phase parasitic coupling are all addressed by considering a reasonable range in the progress of controller design. The realized control law is not implemented as a function of the unmeasurable coupling parameter as has been done in previous works. After a system-variable scaling process for the dynamic normalization and a control input transformation to ensure the controllability of the dynamics in a linear parameter-varying representation, the composite control law is used to address all the inherent uncertainties of the mass, moment of inertia, and parasitic coupling parameter. The composite control law consists of a linear state-feedback gain-scheduled matrix, constructed from parameter-dependent linear matrix inequalities algorithms, and a non-linear control law for thrust manipulation to eliminate the effect of gravitational disturbance. The design is then demonstrated by a time response simulation.

Keywords: vertical/short take-off and landing aircraft, non-minimum phase, robust control, gain scheduling, linear matrix inequality

1 INTRODUCTION

The vertical/short take-off and landing (V/STOL) aircraft has the capability of high mobility and manoeuvrability, and is especially suitable to operate in a poor runway condition such as the deck of a military vessel and an informal temporary runway. The types of V/STOL aircraft, such as the AV-8B Harrier [1] and the F-35 Joint Strike Fighter (JSF) [2], are among the main stream applications at present.

The Harrier is powered by one Rolls Royce Pegasus F402-RR-408 turbo-fan engine with exhaust nozzles equipped in each side of the fuselage to provide the gross thrust for the aircraft. The nozzles are capable of rotating together from the aft position forward approximately 100°. This change in the direction of thrust allows the aircraft to operate in two modes of wing-borne forward flight and jet-borne hovering as well as to transit between them. In order to move in a lateral direction, the aircraft is equipped with reaction control valves in the nose, tails, and wingtips. By using the high pressure flow from the compressor, these valves produce moments around the centre of mass to change the attitude of the roll angle. On the other hand, the JSF is equipped by a modified Pratt and Whitney F119-PW-100 engine configured with shaft-driven lift fan, roll ducts, and a three-bearing swivel main engine nozzle. The lift fan in the front part of frame and the main engine nozzle when swivelled vertically provide the lift force for aircraft manoeuvring in a low air speed. The roll ducts in the both sides of wingtips are used for providing necessary moment for attitude handling.

The hovering operation of the V/STOL aircraft is considered in this paper, in which the upward thrust from the main engine nozzles is manipulated by the throttle for aircraft’s vertical motion. For attitude changing of
the Harrier or JSF aircrafts, if the high pressure air from the reaction control valves or ducts induces an unexpected force component in the lateral direction, the system dynamics will exhibit non-minimum phase characteristics in case the force moves the aircraft into a lateral direction away from the attitude intended. Then, this time delay will cause a non-minimum phase effect on the aircraft dynamics because of the parasitic coupling between input moment and force.

In the recent work by Lin et al. [3], a robustly non-linear state-feedback control law was designed for planar V/STOL aircraft by using an optimal control approach. In reference [4], the non-minimum phase planar V/STOL dynamics were considered as a combination of a linear part resulting from input–output linearization and a non-linear part representing the dynamics that do not depend explicitly on the inputs. Then, a composite non-linear state-feedback control law based on Lyapunov technique with a minimum-norm strategy was developed to stabilize the overall closed-loop system. The authors in reference [5] used a coordinates decoupling technique to deal with this parasitic coupling between input moment and force. Lazano et al. [6] proposed a non-linear controller to achieve globally asymptotical stabilization with emphasis on a real-time application. In reference [7], a finite-preview-based stable-inversion approach for non-linear non-minimum phase system was developed and verified in the planar V/STOL (PVTOL) model.

Another approach for the V/STOL aircraft control is to design a family of controllers beforehand according to the operation envelope of aircraft system. Then, in the real-time application, a mechanism for scheduling among this family of controllers is activated to realize the instantaneous controller based on the aircraft operation. In reference [8], based on the generic V/STOL aircraft model, a robust design using the $\mathcal{H}^\infty$ optimal control technique was investigated. In references [9] and [10], a composite control approach for the robust and robust gain-scheduled design were performed, respectively, for planar V/STOL aircraft dynamics. The formation of composite control law includes a state-feedback gain constructed from linear matrix inequalities (LMIs) algorithm and a non-linear control function to eliminate the effect of gravitational disturbance.

However, it should be noted that in all the approaches [3–7], the control laws were realized as functions of the non-minimum phase parasitic effect and it is undesirable. The non-minimum phase effect is better considered as an uncertain bounded factor, instead of a measurable parameter. On the other hand, though the works in references [9,10] have provided reliable solutions with guaranteed relative stability for the PVTOL aircraft possessing parasitic moment-to-force coupling effect by assuming its possible upper bound in the design algorithms, and also the realized control law is not implemented as a function of this unmeasurable coupling parameter as done previously in the works of references [3] to [7]. The approaches in references [9] and [10] ideally assumed the mass and moment of inertia of the aircraft as constant parameters and then to obtain a normalized PVTOL aircraft dynamics for the subsequent process of controller design.

In this paper, the uncertain properties of the aircraft mass and moment of inertia as well as the parasitic coupling are all addressed by considering their reasonable range in the progress of controller design. The values of the mass and moment of inertia parameters are represented in a linear fractional transformation (LFT) framework [11]. Then, the system variables including the state, control input, disturbance input, and regulated output as well as the parasitic coupling parameter are scaled to obtain normalized PVTOL aircraft dynamics. The non-linear PVTOL dynamics are then formulated as a linear parameter-varying (LPV) system with the roll angle as the system-varying parameter [12,13], in which a control input transformation for the thrust is performed to ensure the controllability of the dynamics in an LPV representation.

The remainder of this paper is organized as follows. Section 2 presents the modelling of a planar V/STOL aircraft dynamics and its reformulation as an LPV system. Section 3 is on the design algorithms for the composite control law. Then, section 4 gives the controller numerical construction and the real-time simulation for the design. Section 5 is the conclusion.

2 PLANAR V/STOL AIRCRAFT DYNAMICS

The states of this PVTOL aircraft include the position of centre of mass, $(X,Y)$, the roll angle, $\theta$, and their corresponding velocities, $(\dot{X}, \dot{Y}, \dot{\theta})$. The control input is the thrust directed to the bottom of aircraft $U_t$ and the moment around the aircraft centre of mass $U_m$. In the case of the air bleeding from the reaction control valves or ducts producing force which is not perpendicular to the pitch axis, there will be a coupling effect between the angle rolling moment and lateral moving force. Let the ratio of lateral force induced by rolling moment be denoted by $\varepsilon_0$, then the aircraft dynamics as shown in Fig. 1 can be written as [3]

\[
\begin{align*}
-m\ddot{X} &= -\sin \theta U_t + \varepsilon_0 \cos \theta U_m \\
-m\ddot{Y} &= \cos \theta U_t + \varepsilon_0 \sin \theta U_m - mg \\
J\ddot{\theta} &= U_m
\end{align*}
\]

(1)

where $mg$ is the gravity force imposed in the aircraft centre of mass and $J$ the moment of inertia around the
The dynamics of the PVTOL aircraft axis through the aircraft centre of mass and along the fuselage.

Let the first and second statements in equation (1) be divided by \( m \), and the third one by \( J \), and the varying quantities of mass and moment of inertia denoted by \( m = m_0(1 + \varepsilon_m) \), \( J = J_0(1 + \varepsilon) \). The PVTOL aircraft dynamics can alternatively be represented by the block diagram as shown in Fig. 2, where the blocks of \( 1/m \) and \( 1/J \) are in the LFT formulation with the matrix parameters

\[
M_1 = \begin{pmatrix} 1/m_0 & -1/m_0 \\ 1 & -1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 1/J_0 & -1/J_0 \\ 1 & -1 \end{pmatrix}
\]

(2)

Introducing the variables \((\ddot{d}_x, \ddot{d}_y, \ddot{d}_z)\) and \((\ddot{z}_x, \ddot{z}_y, \ddot{z}_z)\) in Fig. 2, the PVTOL dynamics in equation (1) can be written as

\[
\begin{align*}
\dot{X} &= -\sin \theta \left( \frac{1}{m_0} U_t - \frac{1}{m_0} \ddot{d}_x \right) \\
&\quad + \cos \theta \left( \frac{1}{m_0} \varepsilon_0 U_m - \frac{1}{m_0} \ddot{d}_y \right) \\
\dot{Y} &= \cos \theta \left( \frac{1}{m_0} U_t - \frac{1}{m_0} \ddot{d}_x \right) \\
&\quad + \sin \theta \left( \frac{1}{m_0} \varepsilon_0 U_m - \frac{1}{m_0} \ddot{d}_y \right) - g \\
\dot{\theta} &= \frac{1}{J_0} U_m - \frac{1}{J_0} \ddot{d}_z 
\end{align*}
\]

(3a)

along with the extra equations derived from the LFT matrices \(M_1, J_1\)

\[
\begin{align*}
\ddot{z}_x &= U_t - \ddot{d}_x \\
\ddot{z}_y &= \varepsilon_0 U_m - \ddot{d}_y \\
\ddot{z}_z &= U_m - \ddot{d}_z
\end{align*}
\]

(3b)

By introducing the following relations for variable normalization

\[
\begin{align*}
-X &= x, & -Y &= y, & \frac{U_t}{m_0 g} &= \varepsilon_1, \\
\frac{U_m}{J_0} &= \varepsilon_2, & \frac{\varepsilon d_0}{m_0 g} &= \varepsilon \\
\frac{\ddot{d}_x}{m_0 g} &= d_x, & \frac{\ddot{d}_y}{m_0 g} &= d_y, & \frac{\ddot{d}_z}{m_0 g} &= d_z \\
\frac{\ddot{z}_x}{m_0 g} &= z_x, & \frac{\ddot{z}_y}{m_0 g} &= z_y, & \frac{\ddot{z}_z}{m_0 g} &= z_z
\end{align*}
\]

The dynamic equations (3a) and (3b) can be written, respectively, as

\[
\begin{align*}
\dot{x} &= -\sin \theta \left( \varepsilon_1 u_1 - d_x \right) + \cos \theta \left( \varepsilon u_2 - d_y \right) \\
\dot{y} &= \cos \theta \left( \varepsilon_1 u_1 - d_x \right) + \sin \theta \left( \varepsilon u_2 - d_y \right) - 1 \\
\dot{\theta} &= u_2 - d_z \\
\ddot{z}_x &= u_1 - d_x \\
\ddot{z}_y &= \varepsilon u_2 - d_y \\
\ddot{z}_z &= u_2 - d_z
\end{align*}
\]

(4a)

(4b)

To obtain a vector–matrix formulation for equations (4a) and (4b) to facilitate the process of controller design, the following vector denotations for state, control input, disturbance input, and regulated output are used

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} := \mathbf{x}, \quad \begin{pmatrix} \varepsilon_1 u_1 \\ \varepsilon_2 \\ \varepsilon d_0 \end{pmatrix} := \mathbf{u}, \quad \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} := \mathbf{d}, \quad \begin{pmatrix} z_x \\ z_y \\ z_z \end{pmatrix} := \mathbf{z}
\]

(5)
Also, denoting the vectors for variables in equations (4a) and (4b) as

\[
\begin{align*}
\mathbf{B}_1(\theta) &:= \begin{pmatrix} 0 & -\sin \theta \\ 0 & \cos \theta \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B}_m &:= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}_c(\theta) := \begin{pmatrix} 0 & \cos \theta \\ 0 & \sin \theta \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathbf{B}_i &:= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_m := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{B}_i := \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\] (6a)

Then, the dynamic equations (4a) and (4b) can be written as,

\[
\begin{align*}
\dot{x}_r &= \mathbf{A}_r x_r + (\mathbf{B}_1(\theta) + \varepsilon \mathbf{B}_2(\theta)) \hat{u} - \mathbf{B}_3(\theta) \mathbf{d} + \mathbf{d}_g \\
\mathbf{z} &= (\hat{\mathbf{B}}_1 + \varepsilon \hat{\mathbf{B}}_2) \hat{u} - \mathbf{d}
\end{align*}
\] (7)

The other matrices in equation (7) are composed of the vectors shown in equations (6a) and (6b) as follows

\[
\begin{align*}
\mathbf{B}_1(\theta) &= (\mathbf{B}_1(\theta) \quad \mathbf{B}_m), \quad \mathbf{B}_2(\theta) = (0 \quad \mathbf{B}_1(\theta)) \\
\mathbf{B}_3(\theta) &= (\mathbf{B}_1(\theta) \quad \mathbf{B}_1(\theta) \quad \mathbf{B}_m) \\
\mathbf{B}_1 &= (\mathbf{B}_1 \quad \mathbf{B}_m), \quad \mathbf{B}_2 = (0 \quad \mathbf{B}_1)
\end{align*}
\] (8)

It is noted that the state dynamics \((x, \dot{x}), (y, \dot{y}), \) and \((\theta, \dot{\theta})\) are decoupled as shown in the system matrix \(A_r\).

Also, the nominal part of the input matrix \(B_1(\theta)\) that is not related to the parasitic parameter \(\varepsilon\), corresponding to the lateral position dynamics \((x, \dot{x})\) is a 2-by-2 zero matrix at the equilibrium condition \(\theta = 0\). Therefore, the LPV representation of the PVTOL aircraft (7) is considered as uncontrollable at the equilibrium \(\theta = 0\) for the lateral position dynamics even in the case of nominal aircraft system with \(\varepsilon = 0\). To remedy this uncontrollability arisen from LPV representation (7) of original non-linear PVTOL dynamics (4), the following correcting procedures are performed.

Let the variable of thrust \(u_t\) be centred around equilibrium point ‘1’, i.e. \(u_1 := 1 + u_t\), and denote \(u := (u_1, u_2)^T\). The first statement in equation (7) can be rewritten as

\[
\dot{x}_r = \mathbf{A}_r x_r + (\mathbf{B}_1(\theta) + \varepsilon \mathbf{B}_2(\theta)) \hat{u} - \mathbf{B}_3(\theta) \mathbf{d} + \mathbf{d}_g + \mathbf{B}_1(\theta)
\]

Denote \(\mathbf{d}_g + \mathbf{B}_1(\theta) := \mathbf{A}_r x_r + \mathbf{d}_g\), where in the matrix \(\mathbf{A}_r\), the only non-zero element is \(\mathbf{A}_r(2, 5) = -1\) and in the vector \(\mathbf{d}_g = (0, -\sin \theta, 0, \cos \theta -1, 0, 0)^T\), the elements ‘\(-\sin \theta\)’ and ‘\(\cos \theta - 1\)’ represent the first-order Taylor series approximation errors of the trigonometric functions ‘\(-\sin \theta\)’ and ‘\(\cos \theta\)’. Then, one can rewrite the first dynamic statement in equation (7) as

\[
\dot{x}_r = \mathbf{A}_r x_r + (\mathbf{B}_1(\theta) + \varepsilon \mathbf{B}_2(\theta)) \hat{u} - \mathbf{B}_3(\theta) \mathbf{d} + \mathbf{d}_g + \mathbf{B}_1(\theta)
\] (9a)

where \(\mathbf{A}_r := \mathbf{A}_r + \mathbf{A}_m\).

Also, by using the notation of the centring control effort \(\hat{u}\), the second statement in equation (7) can be rewritten as

\[
\mathbf{z} = (\hat{\mathbf{B}}_1 + \varepsilon \hat{\mathbf{B}}_2) \hat{u} - \mathbf{d} + \hat{\mathbf{B}}_i
\]

By denoting \(z_x := 1 + z_x'\) and \(z' := (z_x', z_y, z_b)^T\), one can have

\[
\mathbf{z}' = (\hat{\mathbf{B}}_1 + \varepsilon \hat{\mathbf{B}}_2) \hat{u} - \mathbf{d}
\] (9b)

The resulting equations (9a) and (9b) are the LPV representation of the non-linear PVTOL aircraft dynamics and facilitate the controller design in this paper.

3 ROBUST CONTROLLER DESIGN VIA LMI

3.1 Robust gain-scheduled control with uncertain parameters of parasitic coupling, mass, and moment of inertia

Let the normalized variation of mass and moment of inertia be denoted by

\[
\mathbf{Z} := \begin{pmatrix} \delta_m & 0 & 0 \\ 0 & \delta_m & 0 \\ 0 & 0 & \delta_I \end{pmatrix}
\] (10)

then one can have the structure of controller synthesis in this paper as shown in Fig. 3, where an auxiliary disturbance vector \(d' := (d_x', d_y', d_b')^T = \Delta \mathbf{z}'\) is introduced with the multiplicative variation matrix \(\Delta\) which is denoted as

\[
\Delta = \begin{pmatrix} \Delta_m & 0 & 0 \\ 0 & \Delta_m & 0 \\ 0 & 0 & \Delta_I \end{pmatrix}
\] (11)

In the composite control law \(\hat{u} = K(\theta) x_r + \hat{u}_c\), the part of linear parameter-dependent state-feedback control signal \(K(\theta) x_r\) is designed for the PVTOL dynamics (9) without considering the gravitational disturbance \(d_g'\) by using LMIs algorithm, and the part of non-linear control function \(\hat{u}_c\) is designed for the closed-loop system with the linear control signal \(K(\theta) x_r\) connected and the disturbance \(d_g'\) presented.

Consider first the part of linear gain-scheduled control law design by assuming \(\hat{u} = K(\theta) x_r\) with \(d_g' = 0\). Then, after substituting the normalized variation and
control law into aircraft dynamics (9), one can have the closed-loop controlled system

\[
\begin{align*}
\dot{x}_c &= \left(\tilde{A}_o + (B_1(\theta) + \varepsilon B_2(\theta)K(\theta)))\right)x_c - B_3(\theta)\Xi d' \\
&:= A_{cl}(\theta)x_c + B_{cl}(\theta)d' \\
\dot{z}' &= (\tilde{B}_1 + \varepsilon \tilde{B}_2)K(\theta)x_c - \Xi d' := C_{cl}(\theta)x_c + D_{cl}d'
\end{align*}
\]

(12)

where \(\tilde{A}_o = A_o - \lambda I\) with \(\lambda\) the design parameter and \([A_{cl}(\theta), B_{cl}(\theta), C_{cl}(\theta), D_{cl}]\) are the closed-loop system matrices. The state-feedback gain matrix can be constructed by considering a given scaling constant \(\gamma\) for the normalized variation \(\Xi\), then minimized to obtain the parameter \(\lambda^*\). Then, for the controlled system, the maximum relativity stability \(\beta^* = -\lambda^* > 0\) can be established for the quantity of variation \((1/\gamma)\Xi\) on mass and moment of inertia.

Assuming a parameter-dependent Lyapunov matrix \(P(\theta)\) and according to the Bounded-Real Lemma [14], the \(\mathcal{H}_\infty\) performance \(|z'/d|_\infty < \gamma\) if the following matrix inequality holds

\[
\begin{bmatrix}
A_{cl}^T(\theta)P(\theta) + P(\theta)A_{cl}(\theta) + \dot{P}(\theta) & P(\theta)B_{cl}(\theta) \\
B_{cl}^T(\theta)P(\theta) & -\gamma I + D_{cl}^T
\end{bmatrix} < 0
\]

(13)

After substituting the closed-loop system matrices \([A_{cl}(\theta), B_{cl}(\theta), C_{cl}(\theta), D_{cl}]\) from equation (12), denoting \(P^{-1}(\theta) := Q(\theta)\), \(K(\theta)Q(\theta) := L(\theta)\), and the identity \(P^{-1}(\theta)P(\theta)P(\theta)^{-1} = -P(\theta)^{-1} = -Q(\theta)\), one can rewrite the inequality (13) as

\[
\begin{bmatrix}
Q(\theta)A_{cl}^T(\theta) + A_{cl}Q(\theta) + (B_1(\theta) + \varepsilon B_2(\theta))L(\theta) & L^T(\theta) \\
L(\theta)^T(B_1(\theta) + \varepsilon B_2(\theta))^T & -\dot{Q}(\theta) \\
B_2^T(\theta) - \gamma \Xi^{-2} & I \\
(B_1 + \varepsilon B_2)L(\theta) + L(\theta) & -\gamma I
\end{bmatrix} < 0
\]

(14)

Let \(p_c := \sin \theta\), \(p_c := \cos \theta\) and assume the operation range of the roll angle and its variation velocity of the aircraft manoeuvring

\[
|\theta| < \theta_{max}, \quad |\dot{\theta}| < \dot{\theta}_{max}
\]

(15)

then

\[
p_c \in [-\sin \theta_{max}, \sin \theta_{max}] := [p_c, \bar{p}_c] \\
p_c \in [\cos \theta_{max}, 1] := [\bar{p}_c, p_c] \\
\dot{\theta} \in [-\dot{\theta}_{max}, \dot{\theta}_{max}] := [ar{\theta}, \tilde{\theta}]
\]

The parameter-dependent matrix \(B_{cl}(\theta)\) multiplied by the parasitic parameter \(\varepsilon\) can be represented as

\[
B_{cl}(\theta) = (0 B, (p_c, p_c)) \quad := \begin{pmatrix}
p_c - \bar{p}_c & 0 \\
0 & \bar{p}_c
\end{pmatrix}
\begin{pmatrix}
p_c - \bar{p}_c & 0 \\
(\bar{p}_c - p_c)/\bar{p}_c & 1
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]

\[
:= B_{2o} + B_{2r} \Upsilon(p_c, p_c)B_{2r}
\]

(17)

where

\[
B_{2o} = \frac{\bar{p}_c}{\bar{p}_c - p_c} B_{2o} B_{2r} \quad \Upsilon(p_c, p_c) = \begin{pmatrix}
p_c - \bar{p}_c & 0 \\
0 & \bar{p}_c
\end{pmatrix}
\]

(18)

Consider the parameter-dependent matrix \(\Upsilon(p_c, p_c)\), the elements

\[
\Upsilon(p_c, p_c)(1, 1) \in (0, 1), \quad \Upsilon(p_c, p_c)(2, 2) \in [-1, 1]
\]

Therefore, the matrix \(\Upsilon(p_c, p_c)\) satisfies the norm-bounded condition

\[
\Upsilon(p_c, p_c)^T \Upsilon(p_c, p_c) \leq I, \quad \text{for } |\theta| \leq \theta_{max}
\]

(19)

By the application of Schur complement [14], the items \(L^T(\theta)B_{cl}^T(\theta) + B_{cl}(\theta)L(\theta)\) in equation (14) satisfy
the inequality condition

\[
\begin{align*}
L^T(\theta)B_1^T(\theta) + B_2(\theta)L(\theta) \\
= L^T(\theta)B_2^T + \zeta L^T(\theta)B_2L(\theta)
\leq \zeta B_2B_2^T + \zeta^{-1}L^T(\theta)B_2^TB_2L(\theta), \ \forall \zeta > 0 \tag{20}
\end{align*}
\]

Then, one sufficient condition for the matrix inequality (14) can be obtained via Schur complement as

\[
\begin{pmatrix}
Q(\theta)A^T + \tilde{A}_qQ(\theta) \\
+ B_1(\theta)L(\theta) \\
+ L(\theta)B_1(\theta) \\
+ \varepsilon B_2(\theta) \\
+ \varepsilon L(\theta)B_2 \\
+ \varepsilon L(\theta)B_2^T \\
+ \varepsilon B_2B_2^T - Q(\theta) \\
B_2^T(\theta) - \varepsilon^{-1}\varepsilon I \\
B_2(\theta) + \varepsilon B_2^T(\theta)
\end{pmatrix}
\begin{pmatrix}
L^T(\theta)B_2 \\
(\varepsilon I - \varepsilon I) \\
L^T(\theta)B_2 \\
\varepsilon I - \varepsilon I
\end{pmatrix}
\begin{pmatrix}
L^T(\theta)B_2 \\
L^T(\theta)B_2 \\
(\varepsilon I - \varepsilon I)
\end{pmatrix}
\]

\[
< 0 \tag{21}
\]

The parameter-dependent matrices \(B_1(\theta)\) and \(B_2(\theta)\) in equation (14) can be denoted as

\[
\begin{align*}
B_1(\theta) & := B_{10} + p_1B_{1s} + p_cB_{1c}, \\
B_2(\theta) & := B_{20} + p_2B_{2s} + p_cB_{2c}
\end{align*} \tag{22}
\]

with corresponding constant matrices \([B_{10}, B_{1s}, B_{1c}]\) and \([B_{20}, B_{2s}, B_{2c}]\). The parameter-dependent matrix variable \(Q(\theta)\) is assumed to have the same affine dependence on the varying parameters \(p_1, p_2,\) and \(L(\theta)\) assumed to be a single matrix variable to facilitate the numerical tractability of equation (21), that is

\[
Q(\theta) = Q_s + p_1Q_s + p_cQ_c, \quad L(\theta) = L
\]

(23)

Also, by the identity \(Q(\theta) = (p_1Q_s - p_cQ_c)\delta\), the matrix inequality (21) can be represented as functions of the varying parameters \(p_1, p_c, \delta\) and denoted as

\[
\mathcal{L}_o(\zeta, \lambda, \gamma, L, Q_o) + p_1\mathcal{L}_s(\lambda, \hat{\delta}, L, Q_s, Q_c) + p_c\mathcal{L}_c(\lambda, \hat{\delta}, L, Q_c, Q_s) < 0 \tag{24}
\]

with corresponding matrices \([\mathcal{L}_o, \mathcal{L}_s, \mathcal{L}_c]\), which can be evaluated at the vertices \(p_s \in \{p_s^1, \hat{p}_s\}, p_c \in \{p_c^1, \hat{p}_c\}, \delta \in \{-\hat{\delta}, \hat{\delta}\}\) by LMI algorithms and solved for the variables \([\zeta, \lambda, \gamma, L, Q_o, Q_s, Q_c]\). The resulting gainscheduled state-feedback matrix can be obtained as

\[
K(\theta) = L(\theta) + p_1Q_s + p_cQ_c < 0 \tag{25}
\]

with \(K(\theta) < 0\). Then, the condition (24) will be always true for \(\epsilon(t) < \epsilon\).

### 3.2 System constraints on varying parameters and control inputs

The magnitude constraints on control efforts \(\hat{u}_i\), roll angle \(\theta\), and variation rate of roll angle \(\dot{\theta}\) can be addressed by the invariant ellipsoid interpretation \([14, 15]\) on the parameter-dependent quadratic Lyapunov matrix \(P(\theta) > 0\) such that \(\bar{x}_i^T\bar{P}(\theta)\bar{x}_v < 1\) for some initial condition \(x_o\), which in turn can be written as the LMI constraint in the affine parameter-dependent matrix variable \(Q(\theta)\)

\[
\begin{pmatrix}
1 \\
x_v^T \\
Q(\theta)
\end{pmatrix} < 0 \tag{25}
\]

If \(Q(\theta)\) is indeed also a stabilizing solution to the LMI condition described in equation (24), then \(x_i^T\bar{P}(\theta)x < x_i^T\bar{P}(\theta)x_v < 1\). The magnitude constraint \(|\hat{u}_i| < u_{i,max}\) can be satisfied if

\[
\frac{1}{u_{i,\max}^2} - x_i^TQ^{-1}(\theta)L_iQ^{-1}(\theta)x_v < x_i^TQ^{-1}(\theta)x_v
\]

where \(L_i \in \mathbb{R}^{1 \times 6}\) is the \(i\)th row of the matrix \(L\), and can be written as the following LMI condition for given \(u_{i,\max}\)

\[
\begin{pmatrix}
Q(\theta) \\
L^T s_i^T \\
L_i
\end{pmatrix} > 0, \quad i = 1, 2 \tag{26}
\]

where in the row vector \(s_i \in \mathbb{R}^{1 \times 2}\), the \(i\)th element equals 1 and others are zeros.

Similarly, the magnitude constraint on the roll angle, \(|\theta| < \theta_{\max}\), and the variation rate of roll angle, \(|\dot{\theta}| < \dot{\theta}_{\max}\), can be met if the following LMI conditions hold for the given constraints \(\theta_{\max}\) and \(\dot{\theta}_{\max}\)

\[
\begin{pmatrix}
Q(\theta)T \\
\bar{r}_jQ(\theta)T^2
\end{pmatrix} > 0, \quad \begin{pmatrix}
Q(\theta)T \\
\bar{r}_jQ(\theta)T^2
\end{pmatrix} > 0 \tag{27}
\]

where in the row vector \(r_j \in \mathbb{R}^{1 \times 6}\), the \(j\)th element equals 1 and others are zeros.

The matrix inequalities (25) and (26) can be expressed in terms of synthesis variables as

\[
\begin{pmatrix}
L_{u,0}(L, Q_o) + p_1L_{u,s}(Q_s) + p_cL_{u,c}(Q_c) > 0, \quad i = 0, 1, 2 \tag{28}
\end{pmatrix}
\]

Also, equation (27) can be written for controller synthesis as

\[
\begin{pmatrix}
L_{u,0}(Q_o) + p_1L_{u,s}(Q_s) + p_cL_{u,c}(Q_c) > 0 \\
L_{u,0}(Q_o) + p_1L_{u,s}(Q_s) + p_cL_{u,c}(Q_c) > 0
\end{pmatrix} > 0 \tag{29}
\]
3.3 Non-linear control signal for cancellation of gravitational disturbance

Consider the PVTOL aircraft dynamics in an LPV system representation as shown in equations (9a) and (9b), and the controller synthesis structure as shown in Fig. 3. If the design variables obtained from solving the LMI algorithms (24) along with equations (28) and (29) are \([\gamma^*, \lambda^*, L^1, Q^c, Q^e, Q^p]\), by substituting \(d' = (1/y^*) x'\), \(\bar{u} = K^*(\theta) x_c + \hat{u}_c\) into equation (9a), one can solve \(z'\) as

\[
\begin{align*}
\begin{bmatrix} \dot{\bar{u}}_c + (\bar{B}_1 + \varepsilon \bar{B}_2) K^*(\theta) x_c + (\bar{B}_1 + \varepsilon \bar{B}_2) \hat{u}_c \\
- B_3(\theta) \varepsilon I' \left( I + \frac{1}{y^*} Z \right)^{-1} (\bar{B}_1 + \varepsilon \bar{B}_2) K^*(\theta) \\
+ B_1(\theta) + \varepsilon B_2(\theta) - B_3(\theta) \varepsilon Z \left( I + \frac{1}{y^*} Z \right)^{-1} \\
\times (\bar{B}_1 + \varepsilon \bar{B}_2) \end{bmatrix} \dot{x}_c &= \begin{bmatrix} \dot{\bar{u}}_c \\
\bar{B}_3(\theta) \varepsilon \dot{x}_c \\
B_1(\theta) + \varepsilon B_2(\theta) - B_3(\theta) \varepsilon Z \left( I + \frac{1}{y^*} Z \right)^{-1} \\
\times (\bar{B}_1 + \varepsilon \bar{B}_2) \end{bmatrix} \dot{u}_c + d_g \\
&:= \begin{bmatrix} \bar{A}_c x_c + \bar{B}_c \dot{u}_c + d_g \end{bmatrix}
\end{align*}
\]

(31)

Then, from equation (9a), one can have the closed-loop dynamics

\[
\begin{align*}
\dot{z}' &= \left( I + \frac{1}{y^*} Z \right)^{-1} (\bar{B}_1 + \varepsilon \bar{B}_2) K^*(\theta) x_c + (\bar{B}_1 + \varepsilon \bar{B}_2) \hat{u}_c
\end{align*}
\]

(30)

One possibility for \(u_c\) is to simply choose

\[
\begin{align*}
u_c &= (1 + \frac{\delta_m}{\gamma'}) \frac{\sin \theta g_2 + \cos \theta g_1}{\sin \theta + \cos \theta g_2} (34)
\end{align*}
\]

4 SIMULATION

4.1 Controller numerical construction

In this paper, the gain-scheduled robust control for V/STOL aircraft with varying parameters of mass \(m\) and moment of inertia \(J\) is designed according to the specifications of the AV-8B Harrier aircraft. The empty operating weight including only pilot and used fuel of the aircraft is about \(m = 6451\) kg. The maximum full load allowed for a vertical take-off is about 3062 kg, which means a full load operating weight is about \(m = 9521\) kg. Therefore, the normalized variation of the mass depicted in the matrix \(Z\) as shown in equation (10) is computed as \(\delta_m = 0.1918\). Also, by considering the symmetric aircraft frame as a circular cylinder with certain radius, the moment of inertia of the aircraft in the vertical–lateral plane is proportional to the amount of mass. Then, the quantity of the normalized variation of the moment of inertia can be assumed equal to that of the mass as \(\delta_i = \delta_m = 0.1918\).

As for the issue of the control effort capability of the Harrier aircraft dynamics used in this paper, the maximum thrust of the Pegasus engine is \(U_t = 10079\) kg. The minimum operating thrust considered in the hovering operation is assumed equal to the minimum operating weight as \(U_t = m = 6451\) kg. Therefore, the magnitude constraint of the normalized thrust deviation can be computed as \(u_{t,\text{max}} = 0.5624\). The magnitude constraint of the moment \(u_{t,\text{max}}\) is considered as a design parameter with different choices \(u_{t,\text{max}} = (0.3, 0.4, 0.5, 0.6, 0.7, 0.8)\). The magnitude constraints of the dynamic-varying parameters are then chosen as \(|\theta|_{\text{max}} = |\theta|_{\text{max}} = \pi u_{t,\text{max}} k\) with \(k\) as the design parameter. Also, in equation (28), the initial conditions for the states are specified as \(x_c = (\pm 1, 0, 0, 0, 0)\) since the design objective is to keep track of the normalized position command signal in the horizontal and vertical direction.

By evaluating the parameterized LMI s (24), (28), and (29) in the vertices of the parameters \(p_c \in \{p_c^0, p_c\}, p_r \in \{p_r^0, p_r\}, \theta \in [-\theta_{\text{max}}, \theta_{\text{max}}]\) with the previously addressed normalized mass and moment of inertia variation \(Z\), control effort constraints \(u_{t,\text{max}}, u_{t,\text{max}}, \text{and the varying parameters constraints} |\theta|_{\text{max}}, |\theta|_{\text{max}}, \text{and minimizing the stability index} \lambda \in H^\infty \text{ performance level} \gamma = 1 \text{ and parasitic uncertainty} \varepsilon = \{0, 0.05, 0.1, 0.15\}, \) the maximum relative stability \(\beta^* = -\lambda^*\) is obtained and it has been depicted in Fig. 4 for designs with different magnitude constraints on \(u_t, \theta, \text{and} \hat{\theta}\). As expected, the introduced parasitic coupling can degrade the design performance.
4.2 Time response simulation

Two of the constructed controllers will be adopted for performing time response simulation to verify the proposed approach for robust gain-scheduled controller design. One controller is constructed for zero parasitic coupling $\varepsilon = 0$ and the other is designed for $\varepsilon = 0.1$, both with control effort constraints $u_{1,\text{max}} = 0.5624$, $u_{2,\text{max}} = 0.6$ and varying parameters constraints $|\dot{\theta}_{\text{max}}| = |\dot{\theta}_{\text{max}}| = 0.4712$ with $k = 4$. As illustrated in Fig. 4, the achieved maximum relative stability for the cases of $\varepsilon = 0$ and $\varepsilon = 0.1$ are $\beta^* = 0.237$ and 0.233, respectively. These two designs provide guaranteed stability for system dynamics with uncertainty in the normalized mass and moment of inertia $\delta_1 \leq 0.1918$, $\delta_m \leq 0.1918$, and for the case of the parasitic coupling $\varepsilon = 0$ and $\varepsilon \leq 0.1$, respectively. However, since the controllers are constructed based on the developed sufficient condition (21), the controlled system may be able to sustain larger uncertainty than the designed values. The uncertainty in the mass and moment of inertia can include a graduate change caused by fuel consumption and fast variation caused by armament disposal. On the other hand, the uncertainty in the parasitic coupling mainly depends on the accuracy of the mechanism of reaction control valves. In this study, the nature of these uncertainties is characterized by a slow exponential quantity and a portion of fast sinusoid.

In the simulation, reference command signal $(x_r, y_r)$ for the lateral and vertical movement is generated from step command $(x_d, y_d)$ passed through a low-pass filter with bandwidth at 100 rad/s. Figures 5 and 6 are demonstrated for the nominal design with $\varepsilon = 0$, $\delta_m = 0$, $\delta_J = 0$. The varying parameters $\theta, \dot{\theta}$ and the control efforts $u_1, u_2$ are well inside the designated constraints. The lateral–vertical position $x, y$ approaches the command smoothly without values of mass and moment; furthermore, Figs 7 and 8 are demonstrated for the robust design with $\varepsilon = 0.1$. Figure 7 is simulated for an exponential parasitic uncertainty $\varepsilon(t) = 1 - e^{-0.2t}$ and zero uncertainty in mass and moment, whereas Fig. 8 is demonstrated for all the uncertain parameters with ten per cent of sinusoidal fast variation in addition to the exponential uncertainty.

With the step command $(x_d, y_d) = (1, 1)$, the time responses of the states $x, y, \theta, \dot{\theta}$ and the control effort $\tilde{u}_1, \tilde{u}_2$ are shown in Fig. 5 under the specified dynamic parameters $\varepsilon = 0, \delta_m = 0, \delta_J = 0$. The varying parameters $\theta, \dot{\theta}$ and the control efforts $\tilde{u}_1, \tilde{u}_2$ are well inside the designated constraints. The lateral–vertical position $x, y$ approaches the command smoothly without
overshoot. In the case of an exponential parasitic parameter $\varepsilon(t) = 1 - e^{-0.2t}$ being used after $t = 10$ s, an unstable oscillation response can happen to this nominal design as shown in Fig. 6. Therefore, the factor of parasitic coupling needs to be addressed during the phase of controller construction.

On the other hand, if the robust design with $\varepsilon = 0.1$ addressed during controller construction is filled, the time responses under the same conditions as in Fig. 6 are shown in Fig. 7 for considering the same exponential parasitic coupling $\varepsilon(t) = 1 - e^{-0.2t}$ and the nominal values of mass and moment with $\delta_m = 0, \delta_j = 0$. It can be seen that the system can be stabilized and the position states $x, y$ can follow well the given commands, although some minor oscillations are present in the attitude states $\theta, \dot{\theta}$. In addition, if the normalized variations of the mass $\delta_m$, moment of inertia $\delta_j$, and the parasitic coupling are considered as exponential functions with a 10 per cent sinusoidal portion to modeling the fast variation property of the uncertainty parameters, that is $\delta_m(t) = -0.15(1 - e^{-0.5t} + 0.1 \sin t)$, $\delta_j(t) = -0.1(1 - e^{-0.2t} + 0.1 \sin t)$, $\varepsilon(t) = 1 - e^{-0.2t} + 0.1 \sin t$, the time responses under the same command inputs are shown in Fig. 8, where the parameters $\delta_m$ and $\delta_j$ are denoted by dotted line ‘...’ and dash-dotted lines ‘--.--’, respectively. The lateral position tracking is still satisfied with acceptable error between time response $x$ and reference command $x_r$. However, the vertical position tracking exhibits notable tracking error due to the effect of released mass $\delta_m$. Though the quantity of thrust $\tilde{u}_1$ has been suitably reduced, compared with the amount considered in Fig. 7, corresponding to the reduction in mass, the controlled aircraft dynamics still presents excessive altitude response. This discrepancy might be improved in the future by considering the reference command signal during the discussion of controller design algorithm as addressed in section 3.

5 CONCLUSION

This paper has presented a robust gain-scheduled control approach for the planar vertical take-off aircraft dynamics. The uncertain properties of the aircraft mass, moment of inertia, and the non-minimum phase parasitic coupling are all addressed by considering their reasonable range in the progress of controller design. After a system-variable scaling process for the dynamic normalization and a control input transformation to ensure the controllability of the dynamics in an LPV representation, an approach of composite control law is performed addressing all the inherent uncertainties of the mass, moment of inertia, and parasitic coupling parameter. The design is finally demonstrated by various time response simulations, in which the parameter variations for the parasitic coupling, mass, and moment considered exponentially plus a portion of sinusoidal function are presented.

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Appendix

Notation

- \( \mathbf{d}, \mathbf{d}' \): vectors of disturbance input
- \( \mathbf{d}_g, \mathbf{d}_g' \): vectors of gravity disturbance
- \( g \): gravity acceleration constant
- \( J, J_0 \): moment of inertia
- \( K \): state-feedback matrix
- \( m, m_0 \): mass
- \( \mathbf{P} \): Lyapunov matrix
- \( \mathbf{u}, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}' \): vectors of control input
- \( \mathbf{U}_1, \mathbf{u}_1, \tilde{\mathbf{u}}_1, \mathbf{u}_\tau \): thrust force
- \( \mathbf{U}_m, \mathbf{u}_2 \): moment
- \( \mathbf{x}_v \): vector of state
- \( X, x \): lateral position
- \( Y, y \): vertical position
- \( \mathbf{z}, \mathbf{z}' \): vectors of regulated output
- \( \beta, -\lambda \): maximum relativity stability
- \( \gamma \): \( \mathcal{H}^\infty \) performance level
- \( \delta_j \): normalized variation of moment of inertia
- \( \delta_m \): normalized variation of mass
- \( \Delta \): multiplicative variation matrix
- \( \varepsilon_\alpha, \varepsilon \): parasitic coupling between input moment and force
- \( \theta \): roll angle
- \( \Xi \): normalized variation matrix of mass and moment of inertia