Stabilization for a Class of Singularly Perturbed Systems with Multiple Time Delays

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Abstract

In this paper, an asymptotically stabilizing composite feedback control is proposed for a class of linear singularly perturbed systems with multiple time delays. A sufficient condition for the asymptotic stability of the slow subsystem and the fast subsystem is first presented. Moreover, a stability bound $\varepsilon^*$ of singular perturbation parameter $\varepsilon$ is given such that the original system under the composite feedback control is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*)$.

1 Introduction

It is well known that time delay always exists in various engineering systems, such as nuclear reactor, turbojet engines, rolling mills, chemical processes, long transmission lines, pneumatic systems, hydraulic systems, or electric networks. Its existence frequently causes the instability of the system. Therefore, the stability problem of time-delay systems has been a main concern of the researchers over the years; see, for instance, [1-4] and the references therein.

On the other hand, any physical system contains more or less multiple time-scale phenomena. The singular perturbation theory has been proved to be successful in dealing with the multiple time-scale problems; see, for example, [5-11] and the references therein. Consequently, stabilization for the singularly perturbed systems with multiple time delays has been a main concern of researchers [12-15]. If a system is modeled as a singularly perturbed system, then we can use the time-scale properties to separate the time scales and reduce the order of the systems. Furthermore, the dynamics of the...
full-order system can be inferred from those of the reduced-order systems (i.e., the slow subsystem and fast subsystem) when the singular perturbation parameter is sufficiently small. Consequently, it is important to find an upper bound $\varepsilon^*$ of singular perturbation parameter $\varepsilon$ such that stability of the original system can be investigated by establishing that of its corresponding slow and fast subsystems. Thus, if the slow and the fast controls are separately designed such that the slow and the fast subsystems are asymptotically stable, respectively, then the original system under the composite feedback control is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*)$. It is the purpose of this paper to design a composite feedback control to stabilize a class of singularly perturbed systems with multiple time delays.

2 Problem Formulation

For convenience, we define some notation that will be used throughout this paper as follows:

$\bar{\pi} = \{0, 1, 2, \Lambda, n\}$,

$\pi = \{1, 2, \Lambda, n\}$,

$\Re_i$ = set of all nonnegative real numbers,

$C_s$ = right-half s-plane,

$A^{-1}$ = inverse of matrix $A$,

$\lambda_{\text{max}}(A)$ = maximal eigenvalue of matrix $A$,

$\sigma_{\text{max}}(A)$ = largest singular value of matrix $A$;

$$\sigma_{\text{max}}(A) := \left[ \lambda_{\text{max}}(A^*A) \right]^{1/2}.$$

$H(s) \in \mathcal{S}(C_s) = H(s)$ is analytic in $C_s$,

$\|H(s)\|_{\infty} = H_{\infty}$ - norm of $H(s)$,

$\|H(s)\|_{\infty} = \sup_{\Re} \sigma_{\text{max}}[H(jw)]$.

Consider the following singularly perturbed system with multiple non-commensurate time delays:

$$\xi(t) := \sum_{i=0}^{\infty} A_i x(t - h_i) + \sum_{i=\bar{\pi}}^{\infty} A_i z(t - h_i) + B_i u(t), \quad (1a)$$

$$\xi(t) := \sum_{i=0}^{\infty} A_i x(t - h_i) + \sum_{i=\bar{\pi}}^{\infty} A_i z(t - h_i) + B_i u(t), \quad (1b)$$

with $x(t) = \xi(t)$, $z(t) = \phi(t)$, $t \in [-\tau, 0]$, where $A_i$, $A_i$, $A_i$ and $A_i$, $i \in \bar{\pi}$, are constant square matrices with appropriate dimensions and $A_{i0}$ is assumed to be Hurwitz, $h_i = 0$, and $h_i$, $i \in \pi$, are non-negative numbers, $\tau$ is the maximum of $h_i$, $i \in \pi$. Moreover, $\sum_{i=0}^{\infty} A_i$ is assumed to be nonsingular. The small positive scalar $\varepsilon$ in (1b) is a small number, which often occurs naturally due to the presence of small parameters in various physical systems. For instance, it may represent the machine reactance or transient in voltage regulators in a power system model, the time constant of the driver and the actuator in an industrial control system, and it may be due to fast neutrons in a nuclear reactor model.

Remark 1: It is noted that the system (1) is more general than the system considered in [7], where there are no delay terms in the fast state.

According to the time-scale property of the singularly perturbed system [5], the slow subsystem and the fast subsystem of the original system (1) can be derived as follows.

By setting $\varepsilon = 0$, the slow subsystem of (1) is obtained as

$$\xi(t) = \sum_{i=0}^{\infty} A_i x(t - h_i) + \sum_{i=\bar{\pi}}^{\infty} A_i z(t - h_i) + B_i u(t), \quad (2a)$$

$$0 = \sum_{i=0}^{\infty} A_i x(t - h_i) + \sum_{i=\bar{\pi}}^{\infty} A_i z(t - h_i) + B_i u(t), \quad (2b)$$

where $x_i(t)$ and $z_i(t)$ are the slow components of $x(t)$ and $z(t)$, respectively.

If we let $z_i(t) = z_i(t) - z_i(t)$ and $x(t) = x_i(t)$, then (1b) can be rewritten as

$$\tilde{\xi}(t) + \tilde{\xi}(t) = \sum_{i=0}^{\infty} A_i x_i(t - h_i)$$

$$\quad + \sum_{i=\bar{\pi}}^{\infty} A_i [z_i(t - h_i) + z_i(t - h_i)] + B_i u(t). \quad (3)$$

Note that the slow varying state $z_i(t)$ is almost constant with respect to the fast state $z_i(t)$. Therefore, we have $\tilde{\xi}(t) + \tilde{\xi}(t) \approx \tilde{\xi}(t)$. Consequently, the system (3) can be approximated by
\( \mathcal{E}(t) = \sum_{i=1}^{n} A_{i} x_{i}(t-h_{i}) + \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) \\
+ \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) + B_{i} u(t). \) 

(4)

By (2b) and (4), we have

\[ \mathcal{E}(t) = \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) + B_{i}[u(t)-u_{i}(t)] \]

\[ = \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) + B_{i}u_{i}(t), \]

(5a)

where \( u_{i}(t) = u(t) - u_{i}(t) \), i.e., \( u(t) = u_{i}(t) + u_{i}(t) \).

Hence, the dynamics of the fast subsystem (5) is independent of the slow varying states \( x_{i}(t) \) and \( z_{i}(t) \).

**Lemma 1** [7]: If \( H(s) \) is a complex matrix of dimension \( n \times n \), \( H(s) \in S(C_{r}) \), and \( \left\| H(s) \right\| \leq h \), where \( h \) is a constant and \( 0 \leq h < 1 \), then \( [I-H(s)]^{-1} \in S(C_{r}) \) with \( \left\| I - H(s) \right\|^{-1} \leq (1-h)^{-1}. \)

### 3 State Feedback Controls for Slow and Fast Subsystems

In this section, state feedback controls are proposed to stabilize the slow and fast subsystems.

#### 3.1 State Feedback Control for Slow Subsystem

Introducing the slow control

\[ u_{s}(t) = \sum_{i=1}^{m} g_{s_{i}} x_{s_{i}}(t-h_{s_{i}}). \]

(6)

where \( g_{s_{i}}, i \in \mathbb{N} \), are the feedback gains, into the slow subsystem (2) and substituting (6) into (2), we obtain

\[ \mathcal{E}(t) = \sum_{i=1}^{n} A_{i} x_{i}(t-h_{i}) + \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) + B_{i} \sum_{i=1}^{n} g_{s_{i}} x_{s_{i}}(t-h_{i}). \]

\[ = \sum_{i=1}^{n} A_{i} z_{i}(t-h_{i}) + B_{i} \sum_{i=1}^{n} g_{s_{i}} x_{s_{i}}(t-h_{i}). \]

(7)

To neglect the fast mode , we assume that \( z_{i}(t-h_{i}) = \tau(t) \), where \( \tau(t) \) indicates the quasi-steady-state behavior of \( z(t) \) by setting \( \varepsilon = 0 \) in (1). Let

\[ A_{\tau}(s) = \sum_{j=1}^{4} A_{j} e^{-\tau s}, \quad j \in \mathbb{N}. \]

Then we have

\[ \tau(t) = -\sum_{j=1}^{4} A_{j}(0)(A_{j} + B_{j}g_{s})x_{s}(t-h_{j}). \]

(8)

Taking the Laplace transform of (7), we have

\[ sX_{s}(s) = \sum_{i=1}^{n} A_{i} e^{-\tau s} X_{s}(s) + \sum_{i=1}^{n} A_{i} Z(s) + B_{i} \sum_{i=1}^{n} g_{s_{i}} e^{-\tau s} X_{s}(s) + x_{s}(0) \]

(9a)

\[ 0 = \sum_{i=1}^{n} A_{i} e^{-\tau s} X_{s}(s) + \sum_{i=1}^{n} A_{i} \tilde{Z}(s) + B_{i} \sum_{i=1}^{n} g_{s_{i}} e^{-\tau s} X_{s}(s). \]

(9b)

Assume

\[ G_{i}(s) = \sum_{i=1}^{n} g_{s_{i}} e^{-\tau s}. \]

(10)

Substituting (10) into (9b), we obtain

\[ \tilde{Z}(s) = -A_{\tau}(0)X_{s}(s) + B_{i} G_{i}(s)Z(s). \]

(11)

It is obvious that \( A_{\tau}(s) \in S(C_{r}) \) and \( G_{i}(s) \in S(C_{r}) \).

Substituting (11) into (9a), we obtain

\[ X_{s}(s) = M_{s}(s)x_{s}(0), \]

(12a)

where

\[ M_{s}(s) = [sI - A_{\tau}(s) + A_{\tau}(0)A_{\tau}^{-1}(0)A_{\tau}(s) + A_{\tau}(0)A_{\tau}^{-1}(0)B_{i} G_{i}(s)]. \]

(12b)

According to (11) and (12), it is obvious that if \( X_{s}(s) \in S(C_{r}) \), which implies \( \tilde{Z}(s) \in S(C_{r}) \), then the slow feedback subsystem (7) is asymptotically stable.

**Lemma 2**: If we choose \( g_{s_{i}}, i \in \mathbb{N} \), such that

\[ \left\| (sI - A_{\tau}(s))^{-1}[A_{\tau}(s) - A_{\tau}(0)A_{\tau}^{-1}(0)A_{\tau}(s) + [B_{i} - A_{\tau}(0)A_{\tau}^{-1}(0)B_{i} G_{i}(s)] \right\| < 1, \]

(13)

then \( M_{s}(s) \in S(C_{r}) \).

Proof: From (12b), we have

\[ M_{s}(s) = [I - H(s)]^{-1}(sI - A_{\tau}(s)), \]

where

\[ H(s) = (sI - A_{\tau}(s))^{-1}[A_{\tau}(s) - A_{\tau}(0)A_{\tau}^{-1}(0)A_{\tau}(s) + [B_{i} - A_{\tau}(0)A_{\tau}^{-1}(0)B_{i} G_{i}(s)]. \]

If (13) holds, i.e., \( \| H(s) \| < 1 \), then \( [I - H(s)]^{-1} \in S(C_{r}) \) in view of Lemma 1. Moreover, since \( A_{\tau} \) is Hurwitz, we
have \((sI - A_n)\)^{-1} \in S(C_s).\) Thus, we have \(M_f(s) \in S(C_s).\)

This completes the proof.

### 3.2 State Feedback Control for Fast Subsystem

Introducing the fast control

\[
u_f(t) = \sum_{i=0}^{\infty} g_x z_i(t - \tau_i),
\]

(14)

where \(g_x, i \in \mathbb{N},\) are the feedback gains, into the fast subsystem (15) and substituting (14) into (5a), we obtain

\[
e_{\infty}(s) = \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 \sum_{i=0}^{\infty} e^{s \tau_i}(t - \tau_i).
\]

(15)

Taking the Laplace transformation of (15), we have \(e_s Z_f(s) = \sum_{i=0}^{\infty} A_n e^{s \tau_i} Z_f(s) + B_2 \sum_{i=0}^{\infty} e^{s \tau_i} Z_f(s) + \varepsilon_f(0).\) (16)

Let \(G_f(s) = \sum_{i=0}^{\infty} g_x e^{-s \tau_i}.\) Consequently, (16) can be rewritten as

\[
z_f(s) = M_f(s, \varepsilon) \varepsilon_f(0),
\]

(17a)

where

\[
M_f(s, \varepsilon) = \left[ e_s I - A_n(s) - B_2 G_f(s) \right]^{-1}
\]

(17b)

The fast feedback subsystem (15) is asymptotically stable if and only if \(M_f(s, \varepsilon) \in S(C_s).\)

**Lemma 3:** If we choose \(g_x, i \in \mathbb{N},\) such that

\[
\left\| \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right\| < 1
\]

(18)

then \(M_f(s, \varepsilon) \in S(C_s).\)

**Proof:** From (17b), we have

\[
M_f(s, \varepsilon) = \left[ e_s I - A_n - \sum_{i=0}^{\infty} A_n e^{s \tau_i} - B_2 G_f(s) \right]^{-1}
\]

\[
= \left[ e_s I - A_n - \sum_{i=0}^{\infty} A_n e^{s \tau_i} - B_2 G_f(s) \right]^{-1} \left[ e_s I - A_n \right]^{-1}
\]

(19)

By the fact that \(\left\| e_s I - A_n \right\| = \left\| sI - A_n \right\|,\) we have

\[
\left\| e_s I - A_n \right\| \left\| \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right\| \leq \left\| sI - A_n \right\| \left\| \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right\|.
\]

(20)

According to (18) and (20), we obtain

\[
\left\| (sI - A_n)^{-1} \left[ \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right] \right\| < 1.
\]

(21a)

By Lemma 1, the term

\[
\left[ I - (sI - A_n)^{-1} \left[ \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right] \right]^{-1}
\]

in (19) and hence \(M_f(s, \varepsilon) \in S(C_s).\)

This completes the proof.

### 4 Composite Feedback Control

According to \(z_f(t) = z(t) - z_i(t)\) and \(z_i(t - \tau_i) = \tau(t),\)

if we introduce the composite feedback control

\[
u(t) = \sum_{i=0}^{\infty} g_x z_i(t - \tau_i) + \sum_{i=0}^{\infty} g_x z_i(t - \tau_i)
\]

(22)

and substitute (8) into (21), then

\[
u(t) = \sum_{i=0}^{\infty} g_x x(t - \tau_i) + \sum_{i=0}^{\infty} g_x z_i(t - \tau_i)
\]

(23)

Substituting (22) into the original system (1), we obtain

\[
\mathbf{\mathcal{E}}(t) = \sum_{i=0}^{\infty} \tilde{A}_n x(t - \tau_i) + \sum_{i=0}^{\infty} \tilde{A}_n z(t - \tau_i),
\]

(23a)

where

\[
A_n = A_n + B_2 g_x + B_2 \sum_{j=0}^{\infty} g_x A_n^0 \left( A_n + B_2 g_x \right)
\]

(24a)

\[
\tilde{A}_n = A_n + B_2 g_x
\]

(24b)

\[
\tilde{A}_n = A_n + B_2 g_x + B_2 \sum_{j=0}^{\infty} g_x A_n^0 \left( A_n + B_2 g_x \right)
\]

(24c)

Taking the Laplace transform of (23), we have

\[
sX(s) = \sum_{i=0}^{\infty} \tilde{A}_n e^{s \tau_i} X(s) + \sum_{i=0}^{\infty} \tilde{A}_n e^{s \tau_i} Z(s) + x(0),
\]

(24a)

\[
sZ(s) = \sum_{i=0}^{\infty} \tilde{A}_n e^{s \tau_i} X(s) + \sum_{i=0}^{\infty} \tilde{A}_n e^{s \tau_i} Z(s) + \varepsilon(0).
\]

(24b)

Let \(A_j(s) = \sum_{i=0}^{\infty} \tilde{A}_n e^{s \tau_i}, j \in \mathbb{N}.\) Then (24) can be rewritten

\[
\left\| (sI - A_n)^{-1} \left[ \sum_{i=0}^{\infty} A_n e^{s \tau_i} + B_2 G_f(s) \right] \right\| < 1.
\]
as
\[
sX(s) = \tilde{A}_0 X(s) + \tilde{A}_1 Z(s) + x(0),
\]
\[
sZ(s) = \tilde{A}_0 X(s) + \tilde{A}_1 Z(s) + \varepsilon(0).
\]

(25a)

(25b)

\[
A_{\infty} = A_0 + B_1 g_{\infty} \text{ is Hurwitz and the feedback gains } g_{\infty},
\]
\[
i \in \mathbb{N} \text{ and } g_{\infty}, i \in \mathbb{N}, \text{ such that the inequalities (13) and (18) hold, then the original system (1) under the composite feedback control (22) is asymptotically stable for all } \varepsilon \in (0, \varepsilon'). \]

Theorem 1: Choosing the feedback gain \( g_{\infty} \) such that \( A_{\infty} = A_0 + B_1 g_{\infty} \) is Hurwitz and the feedback gains \( g_{\infty}, i \in \mathbb{N} \) and \( g_{\infty}, i \in \mathbb{N}, \) such that the inequalities (13) and (18) hold, then the original system (1) under the composite feedback control (22) is asymptotically stable for all \( \varepsilon \in (0, \varepsilon') \) if the following inequality holds:
\[
\|sI - A_{\infty}\|_\infty \left\| \sum_{i=1}^{\infty} (A_0 + B_1 g_{\infty}) e^{-\varepsilon s} \right\|_\infty = \xi < 1.
\]

(26)

Moreover, a stability bound \( \varepsilon' \) is given by
\[
\varepsilon' = \frac{1 - \xi}{\|s \tilde{M}_1(s) \tilde{A}_1(s) \|_\infty \|sI - A_{\infty}\|_\infty ^2 \| \tilde{A}_1(s) \|_\infty},
\]
with
\[
\tilde{M}_1(s) = \left[ sI - \left[ \tilde{A}_0(s) - \tilde{A}_1(s) \tilde{A}_1^{-1}(s) \tilde{A}_0(s) \right] \right]^{-1}.
\]

(27a)

(27b)

Proof: According to (25) and the results in [8], the system (23) is asymptotically stable for all \( \varepsilon \in (0, \varepsilon') \) if the following inequality holds:
\[
\|\Psi(s, \varepsilon)\|_\infty < 1,
\]

(28a)

where
\[
\Psi(s, \varepsilon) = \tilde{M}_1(s) \tilde{A}_1(s) \tilde{A}_1^{-1}(s) \left[ sI - \tilde{A}_0(s) \right]^{-1} \tilde{A}_0(s).
\]

(28b)

From (28b), we have
\[
\|\Psi(s, \varepsilon)\|_\infty \leq \varepsilon \|s \tilde{M}_1(s) \tilde{A}_1(s) \|_\infty \|sI - A_{\infty}\|_\infty ^2 \| \tilde{A}_1(s) \|_\infty \|
\]

(29)

Moreover, we have
\[
\|sI - \tilde{A}_1(s)\|_\infty ^2
\]
\[
= \left\| I - (sI - \tilde{A}_0) \left[ \sum_{j=1}^{\infty} (A_0 + B_1 g_{\infty}) e^{-\varepsilon j} \right] \right\|_\infty ^2 \left\| sI - \tilde{A}_0 \right\|_\infty ^2 \leq \left\| I - (sI - \tilde{A}_0) \left[ \sum_{j=1}^{\infty} (A_0 + B_1 g_{\infty}) e^{-\varepsilon j} \right] \right\|_\infty ^2 \left\| sI - \tilde{A}_0 \right\|_\infty ^2
\]

(30)

According to Lemma 1, if (26) holds, then
\[
\left\| I - (sI - \tilde{A}_0) \left[ \sum_{j=1}^{\infty} (A_0 + B_1 g_{\infty}) e^{-\varepsilon j} \right] \right\|_\infty ^2 \leq \frac{1}{1 - \xi}.
\]

(31)

From (30) and (31), we have
\[
\|sI - A_{\infty}\|_\infty ^2 \leq \frac{1}{1 - \xi} \left\| sI - A_{\infty} \right\|_\infty ^2.
\]

(32)

According to (27), (29), and (32), if \( \varepsilon < \varepsilon' \), then we have \( \|\Psi(s, \varepsilon)\|_\infty < 1 \). Hence, the composite feedback system (23) is asymptotically stable for all \( \varepsilon \in (0, \varepsilon') \). This completes our proof.

5 Conclusions
A stabilizing composite feedback control has been proposed to guarantee the asymptotic stability of the singularly perturbed systems with multiple time delays. The feedback controls for the slow subsystem and the fast subsystem have been separately designed by frequency-domain stability criteria. Furthermore, a stability bound \( \varepsilon' \) of singular perturbation parameter \( \varepsilon \) has been derived such that the original system under the composite feedback control is asymptotically stable for all \( \varepsilon \in (0, \varepsilon') \).

References


