The Development of Open-loop Optimal Feedback Stochastic Control System

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Abstract—This paper proposes a development methodology to design an estimator for a stochastic system based on a Linear Quadratic Gaussian cost function. Using Merriam’s Expansion, a controller to make system meet the necessity of performance index has been designed. Kalman filter is utilized to derive an estimator for an incomplete state system. The information of the last stage is fed back to design both controller and estimator in the process according to given performance index. The satisfactory results will be easily demonstrated by a single input single output example. The cost function converges to minimum after several stages while the random process is running.

Index Terms—Linear Quadratic Gaussian, Cost Function, Merriam’s Expansion, Random Process

I. INTRODUCTION

Uncertain factors always exist in reality system. How to efficiently and accurately handle uncertain data to describe a physical system keeps going on through last century. Feedback control strategy has been utilized to reduce the corruption of uncertainty originally by Kolmogonov and Wiener since the era of 1940. Stochastic control was then derived to process the random information using probability theory. Fortunately, digital computer and digital signal processor can powerfully afford to process this reality random system with mass information and uncertainty in the recent decades.

The control strategy of a stochastic system depends on what adopted data span from past, present, or predicted stage. They are sorted as the following policies: open-loop policy which uses no any measurement data, open-loop optimal feedback policy which uses current measurement data, the feedback policy which uses all past measurement data and control signal of last stage, m-measurement feedback policy which uses all past measurement data and control signal up to last stage, m measurement feedback policy which uses m stage integrated statistical data in addition to all of last stage, m-measurement feedback policy which uses the all integrated statistical data including measurement data of all stages, control signal up to last stage, and the all integrated statistical data of last stage, and the all integrated statistical data same as predicted m stages integrated statistical data in addition to all past measurement data and control signal of last stage same as predicted m stages integrated statistical data in addition to all of last stage. Open-loop optimal feedback (OLOF) control policy is adopted because of its feasibility to implement in this paper.

Linear state and measurement equation with Quadratic performance index and Gaussian distribution for random variable (LQG) is a popular method to describe a stochastic control system because of neutrality, certainty equivalence, and separability. The linear state equation with plant white noise and measurement equation with measurement noise describe a discrete dynamic stochastic system. In order to efficiently handle this kind of system, the condition of Gaussian distribution for a random variable is compulsory. The quadratic performance index is the basic requirement to meet and the origination to derive a controller that is popular in optimal control methodology. In this paper, one will face at LQG stochastic system and focus on deriving an optimal controller using estimated state in the incomplete state stochastic system.

II. SYSTEM DESCRIPTION

A stochastic dynamic discrete system with state and measurement equation is described as:

\[ X(k+1) = A(k)X(k) + B(k)U(k) + V(k) \]  

(1-1)

\[ Y(k+1) = D(k)X(k) + W(k) \]  

(1-2)

where the mean of the initial state \( X(0) \) is the expectation formulated as

\[ E\{X(0)\} = M(0) \]  

(2-1)

with covariance

\[ COV\{X(0), X^T(0)\} = R(0) \]  

(2-2)

The noise sequence \( V(k), W(k) \) are all have zero mean, i.e.

\[ E\{V(k)\} = 0 \]  

(3-1)

\[ E\{W(k)\} = 0 \]  

(3-2)

and with covariance, independent covariance respectfully expressed as

\[ COV\{V(0), V^T(0)\} = \Sigma_{VV}(0) \]  

(3-3)

\[ COV\{W(0), W^T(0)\} = \Sigma_{WW}(0) \]  

(3-4)

The cost function that indicates performance index is

\[ C_{f}(N) = \sum_{k=0}^{N} \left[ X^T(k)Q_{1}(k)X(k)+U^T(k)Q_{2}(k)U(k) \right] \]  

(4-1)

In order to express the remained cost to be minimized at the \( k^{th} \) stage, cost function always redefined as the form ‘cost-to-go’, that is

\[ C_{f}(N-k) = \sum_{k=0}^{N-1} \left[ X^T(k)Q_{1}(k)X(k)+U^T(k)Q_{2}(k)U(k) \right] \]  

(4-2)

Cost-to-go will not be a definite value except taking
expectation operation of it. Let
\[ J_\text{OLOF}(N-k) = E[\hat{\epsilon}(N-k)] \] (4-3)
where \( J_\text{OLOF} \) is the performance index to be minimized based on OLOF control policy, i.e. it is going to minimize the following conditional mean based on \( Y^{k} \), \( U^{k-1} \),
\[ J_\text{OLOF}(N-k) = \min \{ E[\hat{\epsilon}(N-k)] \} \]
where \( Y^{k} = \{Y(1), Y(2), \ldots, Y(k)\} \) and \( U^{k-1} = \{U(0), U(1), \ldots, U(k-1)\} \)
The task is going to derive an optimal controller to minimize (5-1)

### III. Cost Function Expansion and Controller Derivation

Using Merriam’s parameter expansion\[^8\] one intends to express the related terms of equation between state variable, control signal, noise, and constants, equation (5) then is expressed as
\[ J_\text{OLOF}(N-k) = E[\hat{\epsilon}(N-k)] + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) K(i,j) U(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) N(i,j) V(j) + 2 \sum_{i=1}^{N-1} U(i) f(i) X(k) + 2 \sum_{i=1}^{N-1} V(i) M(i) X(k) \]
where \( a(k) \): positive semi-definiteness constant at \( k \), \( b_k(i) \): \( m \times 1 \) positive semi-definiteness matrix of \( U(i) \), \( K(i,j) \): \( n \times m \) positive definiteness correlation coefficient matrix of \( U(i) \) and \( U(j) \), \( C(k) \): \( n \times 1 \) positive semi-definiteness coefficient matrix of \( X(k) \), \( L(k) \): \( n \times n \) positive semi-definiteness correlation matrix between \( X(k) \) and \( X(k) \), \( g_k(i) \): \( n \times 1 \) positive semi-definiteness matrix of \( V(i) \), \( N_k(i,j) \): \( n \times n \) positive definiteness correlation coefficient matrix of \( V(i) \) and \( V(j) \), \( f_k(j) \): \( n \times n \) positive semi-definiteness coefficient matrix between \( U(j) \) and \( X(k) \), \( M_k(i) \): \( n \times n \) positive semi-definiteness correlation coefficient matrix of \( V(i) \) and \( U(j) \).

Let all the coefficients are independent of \( U^{k-1} \) and boundary conditions are \( L(N) = 0 \), \( a(N) = 0 \). Replace index number \( N-k \) of (6) with \( N-k-1 \), and substitute it into (5), then
\[ J_\text{OLOF}(N-k) = \min \{ E[\hat{\epsilon}(N-k)] \} \]
where
\[ E[\hat{\epsilon}(N-k)] = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) K(i,j) U(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) N(i,j) V(j) + 2 \sum_{i=1}^{N-1} U(i) f(i) X(k) + 2 \sum_{i=1}^{N-1} V(i) M(i) X(k) \]
and
\[ \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) K(i,j) U(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) N(i,j) V(j) + 2 \sum_{i=1}^{N-1} U(i) f(i) X(k) + 2 \sum_{i=1}^{N-1} V(i) M(i) X(k) \]

Substitute equation into (7) and simplify the equation by multiple of substitution and mathematical operation, one obtains
\[ \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) K(i,j) U(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} U(i) N(i,j) V(j) + 2 \sum_{i=1}^{N-1} U(i) f(i) X(k) + 2 \sum_{i=1}^{N-1} V(i) M(i) X(k) \]

One can find there are similar correspondent terms between (6) and (8). Take derivative of (8) with respect to \( U \), then
\[ E[\sum_{i=1}^{N-1} b_k(i) U(i)] + 2 \sum_{i=1}^{N-1} K(i,j) U(j) + 2 \sum_{i=1}^{N-1} f(i) A(i) X(k) + 2 \sum_{i=1}^{N-1} O_k(i,j) V(j) = 0 \]
Rearrange this equation, one gets control equation as follows
\[ E[\sum_{i=1}^{N-1} b_k(i) U(i)] + 2 \sum_{i=1}^{N-1} K(i,j) U(j) + 2 \sum_{i=1}^{N-1} f(i) A(i) X(k) + 2 \sum_{i=1}^{N-1} O_k(i,j) V(j) = 0 \]
If the mean value of noise \( V \) is zero, the optimal controller series is signed as \( U^* \) (**) becomes
\[ \sum_{i=1}^{N-1} U^*[i,j] = \sum_{i=1}^{N-1} [K^*[i,j] f(i) A(i) X(k) + O_k(i,j) V(j) + b_k(i)] \)
Substitute controller into cost-to-go function (8) and take some mathematical operation, then
\[ J_\text{OLOF}(N-k) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} a(i,j) U(i)^T U(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} a(i,j) N(i,j) V(j)^T V(j) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} a(i,j) f(i) X(k)^T X(k) + 2 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} a(i,j) M(i) X(k)^T X(k) \]
where
\[ Q_1(k) = Q_1(k) + A^T(k) L(k) A(k) \]  
(12-1)

\[ A_j(i, j) = A^T(k) f_k^T(i)[K^T_k(i, j)]^{-1} f_k^*(j) A(k) \]  
(12-2)

\[ \sum_{xx} (k|k) = E[(X(k) - \bar{X}(k|k))][X(k) - \bar{X}(k|k)]^T \]  
(12-3)

\[ \sum_{vv} (i, j) = E[(V(i) - \bar{V}(i|k))][V(i) - \bar{V}(i|k)]^T \]  
(12-4)

\[ \sum_{ux} (i, k|k) = E[(V(i) - \bar{V}(i|k))[X(k) - \bar{X}(k|k)]^T \]  
(12-5)

It is essential to estimate incomplete state when disturbance corrupts system. As for a stochastic control system described in (1-1,2), the optimal state estimation will be:

\[ \bar{X}(k+1|k+1) = A\bar{X}(k|k) + BU(k) + K(k+1)[Y(k+1) - \bar{V}(k+1)] \]  
(13)

where Kalman matrix is

\[ K(k+1) = \sum_{xx} (k+1|k) D^T [D \sum_{xx} (k+1|k) D^T + \sum_{ww} (k)]^{-1} \]  
(14)

The covariance of state prediction error is

\[ \sum_{xx} (k+1|k) = A \sum_{xx} (k|k) A^T + \sum_{vv} (k) \]  
(15-1)

and the covariance of state filter error is

\[ \sum_{xx} (k|k) = [I - K(k)D] \sum_{xx} (k|k) \]  
(15-2)

Given initial value of covariance of state filter error

\[ \sum_{xx}(0|0) = \bar{X}_{00}, \text{ initial value of state filtering} \bar{X}(0|0) = \bar{X}_0, \]

one can obtain optimal state estimation using these recursive equations.

### IV. Conclusions

Facing a reality stochastic system, the plant disturbance and measurement noise are un-escapable. One has developed methodology of deriving optimal controller based on a given quadratic performance function accompanied with Kalman filter for an incomplete state system. The random variables are definitely supposed to obey some limitation of zero mean, bound, or being independent of each other. The development results have proposed procedure to simulate and implement a stochastic system by discrete time. The traditional complicated design of an optimal controller has been properly arranged and will be employed easily. Performance index as well as cost function has generalized to the concept of cost-to-go, the cost to be spent.

### References


