Surveys of Discrete-time Variable Structure Control

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Abstract—This paper discussed the recent development of the discrete-time variable structure control (DVSC). The key problems for the study of DVSC include the issues of robustness and the reduction of chattering. The aim of this paper is to provide a brief introduction to the fundamental theory, main results, and practical applications that incorporate this powerful control design approach. DVSC is particularly attractive for the control of nonlinear systems.

Index Terms—discrete-time variable structure control, DVSC, chattering, robustness

I. INTRODUCTION

In recent years, due to its robustness, the applications of variable structure control (VSC) on dynamic systems subject to external disturbances have been extensively studied. Many desirable properties of VSC systems are well documented in references [1-3]. A particular point of interest regarding VSC is the so-called sliding modes. According to the results of previous studies on sliding surface design approaches, the performances of the controlled systems were satisfactory. On the other hand, due to the enormous popularity of microprocessor, the trend of controller design is toward the full digital control. Hence, the need of developing discrete-time variable structure control (DVSC) systems cannot be overlooked.

Over the last 15 years, numbers of papers have been published in the area of DVSC systems. Some of these papers focused on how to map VSC systems into DVSC systems, in which their emphasis is on discretization [4-7]. Other researchers pursued the method of equivalent control, and analyzed new control algorithms based on Lyapunov’s stability theory [8-13]. Their theories were examined for a wide spectrum of system types, including nonlinear systems, MIMO systems, large-scale, infinite dimensional systems and stochastic systems.

Today, benefited from massive and consistent efforts in research and development by numerous researchers, DVSC has been widely applied to various engineering systems. In this introductory article, the basic notion of DVSC and brief discussions about its historical development are presented.

The rest of the paper is organized as follows. Section 2 provides an overview on history, theory and design procedure of DVSC. Present applications of DVSC are contained in Section 3, while the future development of DVSC is included in Section 4. Conclusions are given in Section 5.

II. OVERVIEW OF DVSC THEORY AND DESIGN

Since Utkin proposed the variable structure control (VSC) in 1977 and 1978 [1], VSC has been widely discussed and used in various applications. The control function of VSC is designed to reach the sliding surface that is predetermined to achieve the desired performance. With the advance in the digital computer, the VSC has been extended to the discrete systems.

A. Historic Sketch of DVSC

In the early stage of DVSC, most studies focused on research topics such as SISO systems, linear systems with state feedback gain and switching surface defined in a special quadratic form. During this period, numerous DVSC papers were published and their results were well documented [14-18]. Since 1980, development in the property of perfect robustness of DVSC systems with respect to system perturbation and disturbances [19-22] has greatly enhanced the attention from the research community. As a result, the research and development on DVSC methods have been greatly accelerated, both in theory and applications.

B. Design Procedure

Assume the dynamic equations of the studied system can be expressed as

\[ x_i = x_{i+1}, i = 1, \ldots, n - 1 \]  \hspace{1cm} (1a)

\[ \dot{x}_i = -\sum_{i=1}^{n} a_{ix_i} + bu - f_L \]  \hspace{1cm} (1b)

\[ y = x_i \]  \hspace{1cm} (1c)

Using the forward difference transformation, that is

\[ \dot{x} = \frac{x(k+1) - x(k)}{T} \], one can obtain the corresponding discrete-time dynamic equations as

\[ x_i(k+1) = x_i(k) + Tx_{i+1}(k) \quad i = 1, \ldots, n - 1 \]  \hspace{1cm} (2)

where \( T \) is the sampling time interval. The value of \( T \) is limited by the desired poles in \( Z \)-plane. If one chooses the pole as \( s = -\alpha + j\omega \), then according to the forward difference transformation, one will have

\[ -\alpha + j\omega = \frac{z-1}{T} \]  \hspace{1cm} (3a)

It follows directly from (3a) that the stability condition in \( Z \)-plane is

\[ |z| = \left| -\alpha T + 1 + j\omega T \right| < 1 \]  \hspace{1cm} (3b)

According to (3b), one can obtain the constraint for the
sampling time as
\[ T < \frac{2\alpha}{\omega^2} \]  
(3c)

Let the sliding surface is chosen as
\[ x(k)\sigma(k) = \sum_{i=1}^{n} c_i x_i(k) + x_n(k) = 0 \]  
(4)

In the discrete systems, the control function is computed only at each sampling instance. Therefore, the ideal sliding motion will not exist, that is \( \sigma(k) = 0 \) will not be detected. The non-ideal sliding motion will take place of the ideal sliding motion. The condition for assuring the existence and reachable of the non-ideal sliding motion is
\[ \Delta\sigma(k+1)/\sigma(k) < 0 \]  
(5a)
\[ \Delta\sigma(k+1) \leq \frac{\xi}{2} \]  
(5b)

The objective is to choose the control function to satisfy the eq (5), then a non-ideal sliding motion will appear within a range of \( \xi \). If the final sliding surface is designed to cause the solution of the ideal sliding motion to be asymptotically stable, then the solution of the non-ideal sliding motion will be asymptotically stable, too. Finally, both solutions of the ideal and non-ideal sliding motions are very close within the range of \( \xi \).

Design of such a control function involves several issues. We will discuss those issues in the following:

I. the choice of the control function \( u(k) \) to guarantee the existence of a non-ideal sliding motion;
II. the suppression of chattering of the control function;
III. the determination of the final sliding surface such that the system has the desired properties;
IV. the tuning strategy of the slew rate of the sliding surface.

(I) Choice of the control function

From eq (4), one can have
\[ \Delta\sigma(k+1) = \]  
\[ \sum_{i=1}^{n} a_i x_i(k) + Tb u(k) - TF_L(k) \]  
(6)

where \( a_i \) and \( b \) are the nominal values of \( a_i \) and \( b \), and \( \Delta a_i \) and \( \Delta b \) are the deviations, respectively. Let the control function be constructed as
\[ u(k) = u_q(k) + u_i(k) \]  
(7)

The equivalent control function \( u_q(k) \) is obtained under ideal sliding motion without the parameter variations. Solving eq (6), one obtains
\[ u_q(k) = \left[ -\sum_{i=1}^{n-1} c_i x_{i+1}(k) + \sum_{i=1}^{n} a_i x_i(k) \right]/b^0 \]  
(8)

The switching control function \( u_i(k) \) is used to eliminate the influence of the parameter variation and external load. Let \( u_i(k) \) be formed of
\[ u_i(k) = \sum_{i=1}^{n} \varphi_i x_i(k) + \varphi_{n+1} \]  
(9)

Substitute eq (8)-(9) into eq (7), one obtains
\[ \Delta\sigma(k+1) = \]  
\[ T \sum_{i=1}^{n} \left[ -\Delta a_i - \frac{\Delta b}{b^0} c_i - \frac{\Delta b}{b^0} a_i + b\varphi_i \right] x_i(k) + \]  
\[ T \left[ -f_L(k) + b\varphi_{n+1} \right] \]  
(10)

where \( c_0 = 0 \).

To satisfy existence condition (5a), the limitations on the switching gains are
\[ \sigma^+ < \alpha \Rightarrow \]  
\[ \sup \left[ \left[ \Delta a_i + \frac{\Delta b}{b^0} c_i - \frac{\Delta b}{b^0} a_i \right] /b \right] \]  
if \( \sigma(k)x_i(k) > 0 \)
\[ \sigma^- > \beta \Rightarrow \]  
\[ \inf \left[ \left[ \Delta a_i + \frac{\Delta b}{b^0} c_i - \frac{\Delta b}{b^0} a_i \right] /b \right] \]  
if \( \sigma(k)x_i(k) < 0 \)
\[ i = 1, \ldots, n, \quad c_0 = 0 \]  
(11a)

One can find that the limitations are the same as the ones in continuous systems. The first limitations decide the ranges of the switching gains and divide the choice of the switching gains into two regions.

To satisfy existence condition (5a), the limitations on the switching gains are
\[ T \sum_{i=1}^{n} \left[ -\Delta a_i - \frac{\Delta b}{b^0} c_i - \frac{\Delta b}{b^0} a_i + b\varphi_i \right] x_i(k) + \]  
\[ T \left[ -f_L(k) + b\varphi_{n+1} \right] \]  
(12a)

By setting \( \xi_1 + \cdots + \xi_{n+1} = \frac{\xi}{2} \) where \( \xi_i \) are small positive constants, one can obtain
\[ T x_i(k) = \left[ -\Delta a_i - \frac{\Delta b}{b_0} c_{i-1} + \frac{\Delta b}{b_0} a_i + b \phi_i \right] < \xi_i \]

where

\[ \Phi_i^{lower} = -\max\{|\alpha_i|,|\beta_i|\} \]

and

\[ \Phi_i^{upper} = -\max\{|\alpha_i|,|\beta_i|\} \]

(II) Suppression of chattering phenomena

Due to the sign function, the high frequency switching will give rise to the loss of the energy. The element of controller will be destroyed. Hence, some papers used the saturation function to replace the sign function. It is described as

\[ \text{Sat}(\sigma(k)) = \begin{cases} 1 & \text{if } \sigma(k) \geq \xi \\ \frac{\sigma(k)}{\xi} & \text{if } |\sigma(k)| \leq \xi \\ -1 & \text{if } \sigma(k) \leq -\xi \end{cases} \]

The boundary layer function [21] is proposed in some papers to eliminate the chattering, which is

\[ M_\delta(\sigma(k)) = \frac{\sigma(k)}{\varphi(k) + \delta} \]

(III) Determination of the final sliding surface

As mentioned in the previous section, the sliding surface will decide the desired performance, while the system is exactly locked on the final sliding surface, namely \( \sigma(k) = 0 \). When the motion is locked on this surface, the system’s equations can be reduced and become

\[ x_i(k+1) = x_i(k) + T x_i(k) \quad i = 1, \ldots, n - 2 \]

\[ x_n,k+1 = x_n(k) - T \sum_{i=1}^{n-1} c_i x_i(k) \]

From eq (8), the system’s characteristic equation is

\[ \left( \frac{z-1}{T} \right)^n + c_n \left( \frac{z-1}{T} \right)^{n-1} + \cdots + c_1 \left( \frac{z-1}{T} \right) + c_0 = 0 \]

where \( T \) is the characteristic equation in continuous systems as

\[ s^n + p_n s^{n-1} + \cdots + p_1 s + p_0 = 0 \]

where \( s \) is the Laplace-transform operator. According to forward transform, the pole in s-plane can be transformed to z-plane by

\[ z = \frac{z - 1}{T} \]. Therefore, the coefficients of the final sliding surface can be obtained by comparing eq (19) and (20). One has

\[ c_i = p_i \quad \text{for } i = 1, \ldots, n \]

The system’s order will be reduced. It will increase the stability at the same time.

(IV) Tuning strategy of the slew rate of the sliding surface

The last design procedure is to decide the tuning strategy. Owing to the convergence of the non-ideal sliding motion, one can use the predictive value of sliding motion to detect the convergence. If the predictive sliding motion is convergent into
the small range $\xi$, then the slew rate will be increased till to the desired slew rate.

To determine the coefficients of the sliding surface, the popular approaches including optimal theory [19] and pole placement [15] can be used.

III. APPLICATIONS

A. Motor Control

Control of electrical motors has been a popular application of DVSC. DVSC is proposed for implementing a motor drive system and minimize the computations. Moreover, the chattering phenomenon is eliminated to achieve more smooth and stable control. The following are just a few of many references in the literatures [23-28, 14, 18].

B. Magnetostrictive Control

The primary asset of a DVSC system is the ability to provide high gain feedback while being robust to changes in actuator performance. As magnetostrictive actuators are both hysteretic and time variant [29-30], this response must be linearized. The DVSC strategy is able to linearize an actuator by imposing desired first order response [31-32]. Investigations on the performance of an actual DVSC system were described. The influence of output drive saturation, processor speed, accuracy and finite calculation time on the quality of control was also presented in their study.

C. Other Applications

There are various applications of DVSC to power systems, among these are [33-34]. Their results were later extended by Milosavljevic [35]. Other applications include pulse-width modulation control [36-38], digital implementation [39-40], process control [41-43], DVSC for linear multivariable systems [44-47], piezomechanics control [48-49], plastic extrusion process control [41-43], DVSC for linear multivariable systems [50-52] and a combination of DVSC and deterministic systems [53]. In addition, a nice review paper on recent trends in nonlinear system feedback control can be found in [53]. Undoubtedly, a few other published works of great interest have been missed.

IV. FUTURE DEVELOPMENT

Having a long history of research and development, DVSC is well established as a general control methodology. But, there still are problems to be investigated. For instance, the issues of robustness, chattering and stability deserve more study.

In the DVSC systems can only undergo quasi-sliding modes [21], i.e., the state of the system can approach the switching surface but cannot stay on it, in general. We know, when the state does reach the switching surface, the subsequent discrete-time switching cannot generate the equivalent control to keep the state on the surface. Thus, DVSC systems do not possess the invariance properties found in continuous-time systems; the robustness issues are still under investigation.

In practical systems, however, require an infinitely fast switching mechanism. It is necessary to most DVSC designs. The phenomenon of non-ideal but fast switch was chattering. The high frequency components of the chattering are undesirable. We have to eliminate or reduce it for the DVSC systems.

Obviously, at each slew rate of sliding surface, the control function will make the sliding motion move toward this surface. Once the sliding motion is convergent into the small range, then the new slew rate is obtained. The next step of control function will make the sliding motion convergent to the new sliding surface. Hence, the stability can be analyzed by the classical discrete variable structure systems one slew rate by one.

It is hoped that this account will be of help to those who are interested in understanding the powerful capability of DVSC and in learning how to design DVSC systems.

REFERENCES

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