SOME ISSUES ON CONSISTENCY OF FUZZY ANALYTIC HIERARCHY PROCESS

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Abstract:
This study presents a fuzzy analytic hierarchy process (FAHP) model based on consistent fuzzy linguistic preference relations to deal with consistency. The proposed method yields decision matrices for making pairwise comparisons using additive transitivity. Only \( n-1 \) comparison judgments are required to ensure consistency on a level that contains \( n \) criteria. Finally, a practice example is compared with the proposed method and FAHP method. The proposed method is simple and more accurate than FAHP method.

Keywords:
FAHP; consistent fuzzy linguistic preference relations; consistency; pairwise comparisons

1. Introduction

Most decision-making and problem-solving tasks are too complex to be understood quantitatively. People perform such tasks using knowledge that is imprecise rather than precise. Zadeh [1] introduced the fuzzy set theory in 1965, to rationalize uncertainty associated with impression or vagueness, and thus applicable to human thought. The AHP method can be used to express experts’ opinions, but cannot model human thinking so fuzzy AHP, a fuzzy extension of AHP [2], was developed to solve hierarchical imprecise problems.

The study of consistency is crucial for avoiding misleading solutions. Achieving consistency in Fuzzy AHP is difficult because each positive reciprocal matrix is described by fuzzy numbers. Few investigations have presented a set of mechanisms to solve problems of inconsistency. Furthermore, establishing a fuzzy positive reciprocal matrix requires \( \frac{n \times (n-1)}{2} \) judgments to be made for a level with \( n \) criteria. The numbers of comparisons increases with the number of criteria, so inconsistent conditions are likely to occur.

In order to solve the inconsistency problem, we adopt consistent fuzzy linguistic preference relations to construct fuzzy decision matrix instead of fuzzy positive reciprocal matrix. The proposed method yields decision matrices for making pairwise comparisons using additive transitivity. Only \( n-1 \) comparison judgments are required to ensure consistency on a level that contains \( n \) criteria. Finally, a practice example is compared with the proposed method and FAHP method. The proposed method is simple and more accuracy than fuzzy AHP.

2. Consistent fuzzy linguistic preference relations

The consistent fuzzy linguistic preference relations method was proposed by Wang and Chen [3] to deal with vagueness judgments. This study presents the proposed method to extend on consistency of Fuzzy AHP. The proposed method constructs fuzzy preference relations matrices using fuzzy linguistic assessment variables \( \tilde{p} = (\tilde{p}_{ij}) = (p^L_{ij}, p^M_{ij}, p^R_{ij}) \) based on consistent fuzzy preference relations [4] called consistent fuzzy linguistic preference relations. Table 1 lists the fuzzy linguistic assessment variables.

Table 1. Fuzzy linguistic assessment variables

<table>
<thead>
<tr>
<th>Linguistic variables</th>
<th>Fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Poor (VP)</td>
<td>((0, p^M_{VP}, p^R_{VP}))</td>
</tr>
<tr>
<td>Medium (M)</td>
<td>((p^L_M, 0.5, p^R_M))</td>
</tr>
<tr>
<td>Very Good (VG)</td>
<td>((p^L_{VG}, p^M_{VG}, 1))</td>
</tr>
</tbody>
</table>

Buckley [5] defines the consistency of a fuzzy positive
reciprocal matrix as follows.

**Definition 2.1.** A fuzzy positive matrix \( \tilde{A} = (\tilde{a}_{ij}) \) is reciprocal if and only if \( \tilde{a}_{ij} \otimes \tilde{a}_{kj} = \tilde{1} \), where \( \tilde{1} = (1, 1, 1) \).

**Definition 2.2.** A fuzzy positive reciprocal matrix \( \tilde{A} = (\tilde{a}_{ij}) \) is consistent if and only if \( \tilde{a}_{ij} \otimes \tilde{a}_{kj} = \tilde{a}_{ik} \).

Wang and Chen [3] proposed consistent fuzzy linguistic preference relations. These propositions are described as follows.

**Proposition 2.1.** Suppose a set of alternatives, \( X = \{x_1, \ldots, x_n\} \) associated with a fuzzy reciprocal multiplicative preference matrix \( \tilde{A} = (\tilde{a}_{ij}) \) with \( \tilde{a}_{ij} \in [0,1] \), then the corresponding fuzzy reciprocal linguistic preference relation, \( \tilde{p} = (\tilde{p}_{ij}) = \frac{1}{2} (\log \tilde{a}_{ij} \otimes \tilde{a}_{ji} ) \), verifies the additive reciprocal, namely, the following statements must be equivalent:

1. \( p_{ij}^p + p_{ji}^p = 1 \) \( \forall i, j \in [1, \ldots, n] \).
2. \( p_{ij}^w + p_{ji}^w = 1 \) \( \forall i, j \in [1, \ldots, n] \).
3. \( p_{ij}^w + p_{ji}^w = 1 \) \( \forall i, j \in [1, \ldots, n] \).

**Proposition 2.2.** For a reciprocal fuzzy linguistic preference relation \( \tilde{p} = (\tilde{p}_{ij}) = (p_{ij}^x, p_{ij}^y, p_{ij}^z) \) to be consistent, the following statements must be equivalent:

1. \( p_{ij}^x + p_{ji}^x = \frac{3}{2} \) \( \forall i < j \leq k \).
2. \( p_{ij}^w + p_{ji}^w = \frac{3}{2} \) \( \forall i < j \leq k \).
3. \( p_{ij}^y + p_{ji}^y = \frac{3}{2} \) \( \forall i < j \leq k \).
4. \( p_{ij}^x + p_{ji}^w + p_{ji}^x = \frac{j-i+1}{2} \) \( \forall i < j \).
5. \( p_{ij}^w + p_{ji}^w + p_{ji}^w = \frac{j-i+1}{2} \) \( \forall i < j \).
6. \( p_{ij}^y + p_{ji}^y + p_{ji}^y = \frac{j-i+1}{2} \) \( \forall i < j \).

We note that if the values of the obtained matrix \( \tilde{p} \) with elements \( \tilde{p}_{ij} \) in the interval \([-c,1+c] \) \( (c > 0) \) not in the interval \([0,1]\). In such a case, the obtained fuzzy numbers would need to be transformed using a transformation function which preserves reciprocity and additive consistency, namely a function \( f : [-c,1+c] \rightarrow [0,1] \), verifying

1. \( f(-c) = 0 \).
2. \( f(0+c) = 1 \).
3. \( f(x^w) + f(x^y) = 1 \), \( \forall x \in [-c,1+c] \).
4. \( f(x^y) + f(x^z) = 1 \), \( \forall x \in [-c,1+c] \).

3. Illustrative example

Tolga’s example [6] is used to illustrate the difference between the proposed method and FAHP method. Appendix Fig. 1 displays the hierarchy structure. The Goal is to select the best operating system among the alternatives. The criteria taken into account are Storage Management (C1), Process Management (C2), Protection and Security (C3), Distributed Structure (C4), Software Features (C5), and Financial Figures (C6). The sub-criteria of criterion C1 are Memory Management (C11), Virtual Memory (C12), File System Interface (C13), File System Implementation (C14), and Secondary Storage Structure (C15). The sub-criteria of criterion C2 are Process Handling (C21), CPU Scheduling (C22), Process Coordination (C23), and Deadlocks (C24). The sub-criteria of criterion C3 are Protection (C31), and Security (C32). The sub-criteria of criterion C4 are Network Structures (C41), Distributed System Structures (C42), Distributed File Systems (C43), and Distributed Coordination (C44). The sub-criteria of criterion C5 are Applications and Tools (C51), Bugs and Coding (C52), Graphical User Interface (C53), and Availability and Support (C54). The sub-criterion of criterion C6 is Fuzzy EUAW (C61). Fuzzy linguistic assessment variables are listed in Table 2 by converting Tolga’s Linguistic scale.

<table>
<thead>
<tr>
<th>Linguistic variables</th>
<th>Tolga’s triangle fuzzy numbers</th>
<th>The proposed method’s triangle fuzzy numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demonstrated unimportance (DUI)</td>
<td>(1/3, 2/5, 1/2)</td>
<td>(0, 0.1, 0.2)</td>
</tr>
<tr>
<td>Very strong unimportance (VSUI)</td>
<td>(2/5, 1/2, 2/3)</td>
<td>(0.1, 0.2, 0.3)</td>
</tr>
<tr>
<td>Strong unimportance (SUI)</td>
<td>(1/2, 2/3, 1)</td>
<td>(0.2, 0.3, 0.4)</td>
</tr>
<tr>
<td>Moderate unimportance (MUI)</td>
<td>(2/3, 1, 2)</td>
<td>(0.3, 0.4, 0.5)</td>
</tr>
<tr>
<td>Equal importance (EI)</td>
<td>(1, 1)</td>
<td>(0.4, 0.5, 0.6)</td>
</tr>
<tr>
<td>Moderate importance (MI)</td>
<td>(1/2, 1, 3/2)</td>
<td>(0.5, 0.6, 0.7)</td>
</tr>
<tr>
<td>Strong importance (SI)</td>
<td>(1, 3/2, 2)</td>
<td>(0.6, 0.7, 0.8)</td>
</tr>
<tr>
<td>Very strong importance (VSI)</td>
<td>(3/2, 2, 5/2)</td>
<td>(0.7, 0.8, 0.9)</td>
</tr>
<tr>
<td>Demonstrated importance (DI)</td>
<td>(2, 5/2, 3)</td>
<td>(0.8, 0.9, 1.0)</td>
</tr>
</tbody>
</table>
For example, creating the pairwise comparison matrix of the first level, Table 3 presents the pairwise comparison matrix of the goal, and of all criteria. The weight $W_i$ in Table 3 was determined according to $W_i = \frac{1}{n} \sum_j A_{ij} A_{ij}$ and defuzzification by the equation $\frac{1}{n} \left( p_{ij}^L + p_{ij}^M + p_{ij}^R \right)$ proposed by Yager [7]. The ranking of criteria is C6>C4>C3>C5>C2>C1.

The original data in Table 3 yields the decision matrix of consistent fuzzy linguistic preference relations according to Propositions 2.1 and 2.2. For instance, Table 3 has six criteria. Only $(n - 1 = 6 - 1 = 5)$ values $(p_{12}, p_{23}, p_{34}, p_{45}, p_{56})$ shown on Table 4 are required to construct the decision matrix.

### Table 3. Pairwise comparison of four criteria with respect to the goal

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>(1,1)</td>
<td>(2,3,2)</td>
<td>(2,5,1,2,2,3)</td>
<td>(2,5,1,2,2,3)</td>
<td>(1,2,2,3,1)</td>
<td>(2,5,1,2,2,3)</td>
</tr>
<tr>
<td>C2</td>
<td>(1/2,1,3/2)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
</tr>
<tr>
<td>C3</td>
<td>(2/3,2,5/2)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
</tr>
<tr>
<td>C4</td>
<td>(2/3,2,5/2)</td>
<td>(2,3,1,2)</td>
<td>(1,1)</td>
<td>(2,3,1,2)</td>
<td>(1,1)</td>
<td>(2,3,1,2)</td>
</tr>
<tr>
<td>C5</td>
<td>(1,3/2)</td>
<td>(2/3,1,2)</td>
<td>(1,1)</td>
<td>(2,3,1,2)</td>
<td>(1,1)</td>
<td>(2,3,1,2)</td>
</tr>
<tr>
<td>C6</td>
<td>(3/2,2,5/2)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

### Table 4. Pairwise comparison of four criteria with respect to the goal

<table>
<thead>
<tr>
<th>Criteria</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>C2</td>
<td>x</td>
<td>x</td>
<td>(1,1)</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>C3</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>(1,2,3,2)</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>C4</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>(2,3,1,2)</td>
<td>x</td>
</tr>
<tr>
<td>C5</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>(2,3,1,2)</td>
</tr>
<tr>
<td>C6</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

The whole calculation is as follows:

$p_{31}^L = 1.5 - p_{12}^R - p_{21}^R = 0.7,$

$p_{31}^M = 1.5 - p_{12}^M - p_{21}^M = 0.9,$

$p_{31}^R = 1.5 - p_{12}^L - p_{21}^L = 1.1,$

$p_{41}^L = 2 - p_{12}^R - p_{23}^R = 0.5,$

$p_{41}^M = 2 - p_{12}^M - p_{23}^M = 0.8,$

$p_{41}^R = 2 - p_{12}^L - p_{23}^L = 1.1,$

$p_{42}^L = 1.5 - p_{23}^R - p_{34}^R = 0.2,$

$p_{42}^M = 1.5 - p_{23}^M - p_{34}^M = 0.4,$

$p_{42}^R = 1.5 - p_{23}^L - p_{34}^L = 0.6,$

$p_{51}^L = 2.5 - p_{12}^R - p_{23}^R - p_{34}^R - p_{45}^R = 0.8,$

$p_{51}^M = 2.5 - p_{12}^M - p_{23}^M - p_{34}^M - p_{45}^M = 1.2,$

$p_{51}^R = 2.5 - p_{12}^L - p_{23}^L - p_{34}^L - p_{45}^L = 1.6,$

$p_{52}^L = 2 - p_{23}^R - p_{34}^R - p_{45}^R = 0.5,$

$p_{52}^M = 2 - p_{23}^M - p_{34}^M - p_{45}^M = 0.8,$

$p_{52}^R = 2 - p_{23}^L - p_{34}^L - p_{45}^L = 1.1,$

$p_{53}^L = 1.5 - p_{34}^R - p_{45}^R = 0.6,$

$p_{53}^M = 1.5 - p_{34}^M - p_{45}^M = 0.8,$

$p_{53}^R = 1.5 - p_{34}^L - p_{45}^L = 1.0,$

$p_{54}^L = 3 - p_{23}^R - p_{34}^R - p_{45}^R - p_{56}^R = 1.1,$

$p_{54}^M = 3 - p_{23}^M - p_{34}^M - p_{45}^M - p_{56}^M = 1.6,$

$p_{54}^R = 3 - p_{23}^L - p_{34}^L - p_{45}^L - p_{56}^L = 2.1,$

$p_{55}^L = 2.5 - p_{23}^R - p_{34}^R - p_{45}^R - p_{56}^R = 0.8,$

$p_{55}^M = 2.5 - p_{23}^M - p_{34}^M - p_{45}^M - p_{56}^M = 1.2,$

$p_{55}^R = 2.5 - p_{23}^L - p_{34}^L - p_{45}^L - p_{56}^L = 1.6,$

$p_{56}^L = 2 - p_{23}^R - p_{34}^R - p_{45}^R = 0.9,$

$p_{56}^M = 2 - p_{23}^M - p_{34}^M - p_{45}^M = 1.2,$

$p_{56}^R = 2 - p_{23}^L - p_{34}^L - p_{45}^L = 1.5,$

$p_{64}^L = 1.5 - p_{45}^R - p_{56}^R = 1.1,$

$p_{64}^M = 1.5 - p_{45}^M - p_{56}^M = 1.3,$

$p_{64}^R = 1.5 - p_{45}^L - p_{56}^L = 1.5,$

$p_{65}^L = 1 - p_{45}^R = 0.4,$

$p_{65}^M = 1 - p_{45}^M = 0.5,$

$p_{65}^R = 1 - p_{45}^L = 0.6,$

$p_{66}^L = 1 - p_{45}^R = 0.3,$

$p_{66}^M = 1 - p_{45}^M = 0.4,$

$p_{66}^R = 1 - p_{45}^L = 0.5,$
are 0.5, 0.69) 0.34, 0.38, 0.41) 0.41, 0.47, 0.53) 0.47, 0.5, 0.53) 0.5, 0.59, 0.69) 0.59, 0.62, 0.69) 0.62, 0.69, 0.75) 0.69, 0.75, 0.81). 

The result obtained from the FAHP method and the proposed method is different. That's because it didn't check consistency in the FAHP method. For instance, C1 is Moderate unimportance than C2 and C2 is Equal importance than C3, according transitivity then C1 should be Moderate unimportance than C2, but in Tolga’s example, C1 is very strong unimportance than C3, it is somewhat unreasonable. Due to the proposed method only n-1 (P12, P23, ..., P(n-1)n) comparison times for n criteria, so inconsistent conditions are not to occur. Thus, the proposed method is simple and more accuracy than FAHP method.

4. Conclusions

This study applies consistent fuzzy linguistic preference relations to construct fuzzy decision matrices. The proposed method can allow decision-makers to meet imprecise or vague environment, deal with the problem of Fuzzy AHP consistency and only requires n-1 comparison judgments constructing fuzzy decision matrices. The illustrated example involves six criteria needed five comparison judgments. Therefore, the numbers of pairwise comparisons can be reduced by \((C_6^6 - 5) = 10\) times while still ensuring consistency. The numbers of pairwise comparisons can be reduced by \((C_5^5 - 5) + (C_4^4 - 4) + 3 	imes (C_3^3 - 3) = 25\) times for the whole hierarchy. The proposed method enhances decision efficiency and accuracy. The increase in the number of criteria and alternatives reduces the number of comparisons required by the proposed method. This study provides a set of mechanisms to derive consistency fuzzy rankings.

References

Appendix.

![Fig. 1. The hierarchy structure]