Analysis of Z-branch from a pre-existing slipping crack in an anisotropic solid

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Abstract

The problem of Z-branch from a pre-existing slipping crack in an anisotropic solid is formulated in terms of a system of singular integral equations, which is developed by way of Stroh formalism. The frictional forces existing on the slipping crack are directly incorporated into the formulation. A numerical method is then employed to solve these equations. Two points are addressed in the present investigation. First, the problem of an infinitesimally small branched crack length is investigated, in which the effects of frictional forces on the stress intensity factors at the branched crack tips are studied. Then the level curves for the stress intensity factors and energy release rate are given, which are plotted versus various branch lengths and branch angles for the case of frictionless slipping surface problem. From the level curves some phenomena are observed and results are discussed.

Keywords: Z-branch; Slipping crack; Anisotropic solid; Stroh formalism; Stress intensity factor; Energy release rate

1. Introduction

The study of branched crack extension from a pre-existing slipping crack is of fundamental importance in the field of fracture mechanics, and much progress has been made for isotropic materials. One way to deal with branched crack problems is to assume a finite length of branched crack that already exists and then analyze the near tip fields around the tip of the branched crack. Such analyses include the works of, e.g., Cotterell [8], Hoek and Bieniawski [11], McClintock and Walsh [19], Okubo and Peng [24], Sih [25,26], as well as Swedlow [31]. Another way of dealing with branched crack problems is to assume that the length of the branched crack is infinitesimally small and then the most probable direction of crack branching is analyzed. The work of Aravas and McMee-

king [1] is somewhat along this line. In addition to the above two ways of approach, there are other works which assume that the branched crack grows initially from an infinitesimally small length and then continues to grow in a fixed direction to a finite length. Analyses of this kind include Brace and Bombolakis [6], Horii and Nemat-Nasser [12,13], Horii [14], Krajcinovic et al. [17], as well as and Nemat-Nasser and Horii [22]. A comprehensive summary of the crack kinking and curving problems has been provided by Karihaloo [16]. All the above investigations are for isotropic materials.

Anisotropic problems have also been considered by many investigators. Most of them, e.g., Azhdari and Nemat-Nasser [2], Blanco et al. [4,5], Ichikawa and Tanaku [15], Maiti [18], Miller and Stock [20], Obata et al. [23], as well as Sih [27] etc., deal with problems that are subjected to tensile loadings at infinity. Hence, conditions on the crack faces are assumed to be traction-free. In the present analysis, we will consider the problem of the closure of the main crack. However, the slipping of the main crack is allowed when the magnitude of the
far field compressive loadings reach certain values. An antisymmetric branched crack, called Z-branch, is assumed to branch simultaneously at both crack tips of the main crack (Fig. 1). The faces of the branched crack are assumed to be open when the far field compressive loadings are applied. However, the weaker singularity at the kink itself is assumed in the present analysis. With Stroh formalism, such a problem is formulated in terms of a system of singular integral equations. Effects of frictional forces developed on the main crack are directly incorporated into the formulation. A numerical method is then employed to solve these coupled singular integral equations from which the stress intensity factors and the corresponding energy release rate are determined directly. Two points are addressed in the present analysis. First, the problem of an infinitesimally small branched crack length is analyzed wherein the effect of the coefficient of friction $\mu$, on the stress intensity factors is studied. The stress intensity factors at the branched crack tips are computed for various values of $\mu$. Our results show that stress intensity factors decrease as the magnitude of $\mu$ increases, an expected behavior intuitively. Next, we investigate the behavior of the stress intensity factors and energy release rate for various normalized branch lengths ($\ell/c$, ratio of branched length to half of the length of the main crack) and branch angles ($\theta$) assuming the absence of frictional forces (i.e., frictionless slipping surface is considered). Level curves are plotted for both stress intensity factors and energy release rate. From level curves we observe that there exits flat region where $K_t$ is insensitive to $\theta$ and $\ell/c$. Also observed from level curves is that $K_t$ has the tendency to remain constant when branched crack grows long enough. The phenomena of $K_{H1}$, different from those for $K_t$, are also discussed. Level curves for energy release rate, $G$, reveal that there exist saddle points for $G$. The usefulness of these level curves for $G$ is also discussed.

2. Basic equations

A two-dimensional deformation of a linear elastic solid whose field quantities are only functions of $x_1$ and $x_2$ is considered. The general expressions for the displacement $u$ and stress function $\phi$ for such a deformation are [9,28]

\[
\begin{align*}
\mathbf{u} &= 2\text{Re}\{\mathbf{A}f(z)\} \\
\phi &= 2\text{Re}\{\mathbf{B}f(z)\}
\end{align*}
\]

where $\text{Re}\{\cdot\}$ denotes real part,

\[
f(z) = (f_1(z_1), f_2(z_2), f_3(z_3))^T
\]

with $z_k = x_1 + p_kx_2$ ($k = 1, 2, 3$). Superscript ‘T’ represents transpose. Matrix $\mathbf{A}$ with components denoted by $a_{ij}$ and constants $p_k$ are determined from the following eigenvalue problem

\[
\{c_{ijkl} + p_j(c_{i1k2} + c_{i2k1}) + p_j^2c_{i3k2}\}a_{ij} = 0 \quad \text{(no sum on $j$)}
\]

where $c_{ijkl}$ are the elastic constants. Without loss of generality, one may take the imaginary part of $p_k$ to be positive. Matrix $\mathbf{B}$ in Eq. (2) is defined by

\[
\mathbf{B} = \mathbf{R}^T\mathbf{A} + \mathbf{TAP}
\]

where

\[
R_{ik} = c_{i1k2}, \quad T_{ik} = c_{i2k1}
\]

and $\mathbf{P} = \text{diag}(p_1, p_2, p_3)$. (For more detail description of the function $f(z)$ and the physical meaning of matrices $\mathbf{A}$ and $\mathbf{B}$, please refer to the paper by, e.g., Ting [32]). Stress function $\phi$ are related to the stress components by

\[
\begin{align*}
t_1 &= (\sigma_{11}, \sigma_{12}, \sigma_{13})^T = -\frac{\partial\phi}{\partial x_3} = -2\text{Re}\{\mathbf{BPf}'(z)\} \\
t_2 &= (\sigma_{21}, \sigma_{22}, \sigma_{23})^T = \frac{\partial\phi}{\partial x_1} = 2\text{Re}\{\mathbf{Bf}'(z)\}
\end{align*}
\]

where

\[
f'(z) = \frac{df(z)}{dz} = \left(\frac{df_1(z_1)}{dz_1}, \frac{df_2(z_2)}{dz_2}, \frac{df_3(z_3)}{dz_3}\right)^T
\]

A more general expression (see, e.g. [32])

\[
t_n = \frac{\partial\phi}{\partial \xi^n}
\]

can be established where $s$ is the arc length and $\mathbf{n}$ the unit outward normal vector. Matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following orthogonality relations [28,7]:

\[
\begin{bmatrix}
\mathbf{A} & \bar{\mathbf{A}} \\
\mathbf{B} & \bar{\mathbf{B}}
\end{bmatrix}^T = \begin{bmatrix}
\mathbf{I} & 0 \\
0 & \mathbf{I}
\end{bmatrix}
\]

where $\mathbf{I}$ is a $3 \times 3$ unit matrix and a bar over a quantity represents the conjugate of that quantity.

Fig. 1. Z-branch from a pre-existing slipping main crack.
3. Problem formulation

3.1. Problem description

Consider the problem of a Z-shaped crack that is embedded in a 2-D infinite elastic solid as depicted in Fig. 1. The Z-shaped crack consists of a main crack PP’ with length 2c and of two branched cracks that are characterized by a length ℓ and an angle parameter θ. The stresses applied at infinity in the x₁ and x₂ directions are denoted by \( r_1^∞ = [σ_{11}, σ_{12}, σ_{13}]^T \) and \( r_2^∞ = [σ_{21}, σ_{22}, σ_{23}]^T \), respectively. \( σ_{21}^+ \) and \( σ_{22}^- \) are assumed to be always negative. The surface of the main crack is closed but is capable of transmitting frictional force (denoted by \( μσ_{22} \); \( μ \), the frictional coefficient, is assumed to be constant in the analysis). According to the above description, the boundary conditions on the main crack are

\[
\begin{align*}
 u_1^+ &= u_2^+, & u_1^- &= u_2^- \\
 σ_{21}^+ &= σ_{21}^- = μσ_{22}, & |x_1| < c, & x_2 = 0
\end{align*}
\]

where \( u_2 \) and \( u_3 \) are the displacements in the \( x_2 \) and \( x_3 \) directions, respectively. The superscript plus (minus) denotes the quantity that is on the upper (lower) crack face. Since the two branched cracks are considered to be tensile cracks, we impose traction free conditions along the branched cracks, which are

\[
r'_η = [σ_{η1}, σ_{η2}, σ_{η3}]^T = Ω(θ)t_η = 0, \quad 0 < ξ < ℓ, \quad η = 0^- \tag{14}
\]

where \( r'_η \) is the traction expressed in terms of the local coordinates \((ξ, η, x_3)\), its origin is centered at \( x_1 = c \) (or \( x_1 = -c \)) with the direction of \( ξ \) being parallel to the face of the branched crack (see Fig. 1), whereas \( t_η \), as defined by Eq. (10), is the traction (which is zero) that is expressed in terms of global coordinates \((x_1, x_2, x_3)\). In terms of the local coordinates, one can interpret \( σ_{η2} \) as the normal stress while \( σ_{η1} \) and \( σ_{η3} \) as the shear stresses acting on the extended branched crack faces. In Eq. (14), the \( 3 \times 3 \) matrix \( Ω(θ) \) whose components are cosine of the angle between the local and global coordinates is given by

\[
Ω(θ) = \begin{bmatrix}
\cos(θ) & \sin(θ) & 0 \\
-\sin(θ) & \cos(θ) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

where \( θ \) is defined positive when measured from the positive direction of \( x_1 \)-axis to that of \( ξ \)-axis, as shown in Fig. 1.

3.2. Singular integral equations

The problem described above can be formulated in terms of a system of singular integral equations. Since the formulation for anisotropic material is known to be parallel with that for isotropic materials, as thoroughly presents in previous papers, e.g. [16] and the references cited therein, therefore in the following we will briefly outline the formulation for anisotropic material. For more details please refer to the paper by Kar-upaloo [16]. First we develop the stress function \( ϕ \) such that the boundary condition stated in Eq. (13) and that the far field applied loading condition are both satisfied. The stress function \( ϕ \) can be determined by superposition, i.e., let \( ϕ = ϕ^D + ϕ^R \). The stress function \( ϕ^D \) is due to a pair of dislocations, each located anti-symmetrically on the branched cracks, with Burgers vectors denoted by \( b^*= [b_1^*, b_2^*, b_3^*]^T \) and \( -b^* \), respectively (see Fig. 2). The stress function \( ϕ^D \) of the corresponding function \( f_D(z) \) is developed in such a manner that the conditions on the surface of the main crack listed in Eq. (19) are satisfied and its development can be achieved by separating it into two parts as follows

\[
ϕ^D = ϕ^d + ϕ^R \tag{16}
\]

where \( ϕ^d \) is the stress function that is due to the pair of dislocations acting in an infinite body which is totally free of any cracks while \( ϕ^R \) is adding so that the conditions of (19) for \( ϕ^D \) are satisfied. The stress function \( ϕ^d \) is given by [9,30]

\[
ϕ^d = 2Re \{B_{ij}(z)\} \tag{17}
\]

where

\[
f_D(z) = \text{diag} \left[ \ln \left( \frac{z_1 - z_1' }{z_1 + z_1'} \right), \ln \left( \frac{z_2 - z_2' }{z_2 + z_2'} \right), \ln \left( \frac{z_3 - z_3' }{z_3 + z_3'} \right) \right] B' \left( \frac{b^*}{2m} \right)
\]

in which \( z_k = x_1 + p_kx_2 \), and \( z_3' = x_1^D + p_kx_2^D \) \((k = 1, 2, 3)\). Here \((x_1^D, x_2^D)\) and \((-x_1^D, -x_2^D)\) are points where point dislocations are acting. On the main crack surface, the stress function \( ϕ^D \) must satisfy

\[
r_2^D(x_1) = r_1^D(x_1) + r_2^D(x_1) = (I - R)r_1^D(x_1), \quad |x_1| < c, \quad x_2 = 0
\]

\[
\text{Fig. 2. A pair of dislocations acting at points } z_0 \text{ and } -z_0, \text{ respectively, with conditions on the main crack listed in Eq. (19).}
\]
where \( \mathbf{T}^D = [\sigma_{21}^D, \sigma_{22}^D, \sigma_{23}^D]^T \) is the traction induced by \( \phi^D \) and
\[
\mathbf{R} = \begin{bmatrix}
1 & -\mu & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]  
(20)
in which \( \mathbf{I} \) is a 3 \( \times \) 3 unit matrix; \( \mathbf{T}^S = [\sigma_{21}^S, \sigma_{22}^S, \sigma_{23}^S]^T \) is the traction produced by \( \phi^S \); and while \( \mathbf{T}^D = [\sigma_{21}^D, \sigma_{22}^D, \sigma_{23}^D]^T \) is that due to \( \phi^D \). Since \( \phi^S \) is known (Eqs. (17) and (18)), therefore, on the main crack the traction induced by \( \phi^S \) can be evaluated as
\[
\mathbf{T}^S(x_1) = 2 \text{Re} \left\{ \mathbf{R} \text{diag} \left( \frac{z_1^D - z_1^S}{x_1^2 - x_1^D}, \frac{x_1^S - x_1^D}{x_1^2 - x_1^S}, \frac{z_1^D - z_1^S}{x_1^2 - x_1^S} \right) \right\} \mathbf{B} \left( \frac{b^S}{\pi} \right)
\]
where \( |x_1| < c \)
(21)
Hence, from Eq. (19), the traction induced by \( \phi^R \) must satisfy
\[
\mathbf{T}^R(x_1) = [\sigma_{21}^R, \sigma_{22}^R, \sigma_{23}^R]^T = -\mathbf{R} \mathbf{T}^S(x_1), \quad |x_1| < c, \quad x_2 = 0
\]
(22)
The procedure of finding \( \phi^R \) (or the corresponding function \( f_3(z) \)) is a typical Hilbert problem (Muskhelishvili [21]). By adding the function \( f'_3(z) \) (Eq. (18)) and \( f'_0(z) \), then the function \( f'_3(z) \) corresponding to stress function \( \phi^D \) may be easily obtained. Both functions are given in Appendix A. Following the above procedures, the stress function corresponding to the external forces applied at infinity which is denoted by \( \phi^S \) can be obtained by first separating \( \phi^S \) as the superposition of two stress functions, i.e.,
\[
\phi^S = \phi^\infty + \phi^{\infty R}
\]
(23)
where \( \phi^\infty = x_1 t_2^\infty - x_2 t_1^\infty \) is the stress function due to external forces acting at infinity while the solid is free of any crack and the stress function \( \phi^{\infty R} \) is added so that \( \phi^S \) will satisfy the following conditions on the main crack, i.e.,
\[
\mathbf{T}^R = \mathbf{t}^\infty + \mathbf{t}^{\infty R} = (\mathbf{I} - \mathbf{R}) \mathbf{t}^\infty, \quad |x_1| < c, \quad x_2 = 0
\]
(24)
where \( \mathbf{t}^{\infty} = [\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}]^T \) is the traction induced by \( \phi^\infty \) and \( \mathbf{t}^{\infty R} = [\sigma_{21}^{\infty R}, \sigma_{22}^{\infty R}, \sigma_{23}^{\infty R}]^T \) is that due to \( \phi^{\infty R} \). The determination of \( \phi^{\infty R} \) is again a Hilbert problem and the result is
\[
\phi^{\infty R} = 2 \text{Re} \{ \mathbf{B} \mathbf{f}_3(z) \}
\]
(25)
where
\[
f'_3(z) = -\frac{1}{2} \text{diag} \left( (z_1 \chi(z_1) - 1), (z_2 \chi(z_2) - 1), (z_3 \chi(z_3) - 1) \right) \mathbf{B}^{-1} \mathbf{t}^{\infty}
\]
(26)
and
\[
\mathbf{t}^{\infty} = -\mathbf{R} \mathbf{t}^{\infty}
\]
(27)
Since the constructions of \( \phi^D \) and \( \phi^F \) are such that the conditions in Eqs. (19) and (24) are satisfied separately, therefore, when adding \( \phi^D \) and \( \phi^F \) together, one would find that the obtained stress function \( \phi = \phi^D + \phi^F \) will satisfy not only the condition listed in Eq. (13) but also the far field condition applied at infinity.

Now suitably distributed dislocations are introduced to represent the branched extension of the crack. Let \( \mathbf{b}^{\infty}(t) = \mathbf{b}(t) \mathbf{d} \), where \( \mathbf{b}(t) \) is the dislocation density. Since the conditions on the pre-existing main crack are automatically satisfied, only the traction free conditions (14) on the branched crack extension remain to be enforced. By doing so, a system of singular integral equations is obtained which is
\[
-\frac{1}{2\pi} \left\{ \mathbf{L} \int_0^\ell b(t) \frac{\mathbf{d}}{r - \xi} \, dt + \int_0^\ell K(\xi, t) b(t) \, dt \right\} = g(\xi), \quad 0 < \xi < \ell
\]
(28)
where matrix \( \mathbf{L} = -2i \mathbf{B}^T \mathbf{B} \) and the explicit expressions of the kernel \( K(\xi, t) \) and the vector function \( g(\xi) \) are given in Appendix A. For single-valued displacements of \( u_2 \) and \( u_3 \) (see Eq. (12)), the following auxiliary conditions must be satisfied
\[
\int_0^\ell b_k(t) \, dt = 0, \quad k = 2, 3
\]
(29)

### 3.3. Stress intensity factors and energy release rate

The coupled singular integral equations for the dislocation densities shown in Eq. (28) combined with Eq. (29) can be solved numerically. Once the dislocation density functions have been found, one can then compute the stress intensity factors at the crack tips by the formula [29]
\[
k = [K_{II}, K_{I}, K_{III}]^T = \sqrt{\frac{\pi}{2\ell}} \mathbf{L} \mathbf{b}(\ell)
\]
(30)
where \( \mathbf{b}(\ell) \) is the value of \( \mathbf{b}(t) \) at \( t = \ell \) and \( \mathbf{b}(\ell) \) is related to \( \mathbf{b}(t) \) by
\[
\mathbf{b}(t) = \frac{\mathbf{b}(\ell)}{\sqrt{\ell(\ell-t)}}, \quad 0 < t < \ell
\]
(31)
where the dislocation densities having square root singular at the branched crack tip have been implied. The expression for the energy release rate \( G \) has been given by Branett and Asaro [3]. It is related to the stress intensity factors by
\[
G = \frac{1}{2} \mathbf{K}^T \left( \mathbf{\Omega}(\theta) \mathbf{L}^{-1} \mathbf{\Omega}^T(\theta) \right) \mathbf{k}
\]
(32)
for general anisotropic materials. Substituting (30) into above equation, one obtains
\[
G = \frac{\pi}{4\ell} \mathbf{b}(\ell)^T \mathbf{L} \mathbf{b}(\ell)
\]
(33)

### 4. Numerical results and discussions

#### 4.1. Preliminaries

The coupled singular integral equation (28) with (29) can be solved numerically using a method suggested by Gerasoulis [10]. The method uses the piecewise quadratic polynomials to approximate the continuous functions \( \mathbf{b}(t) \)
defined in Eq. (31). The stress intensity factors at the tips and energy release rate are then obtained from Eqs. (30) and (33), respectively. For the present problem, the displacement of \( u_1 \) on the main crack is allowed to slide, therefore, the singular behavior for stresses at knee of the kinked crack will be less than one half [20,23]. Based on this assumption, we then let \( b_1(0) = 0 \) in the numerical computations. Although the derivation in Section 3 applies to 2-D general anisotropic material with fifteen independent elastic constants, our numerical results are carried out for orthotropic materials only. For simplicity, we further assume that the material alignments are such that the \( x_3 \)-axis is an axis of material symmetry. This assumption essentially decouples the antiplane behavior from the inplane response, and henceforth we will ignore the antiplane problem. In our numerical calculation, the principal compliances are taken to be as follows: \( E_2 = E_3 = E_1/\beta \), \( v_{12} = v_{13} = v_{23} = v = 0.25 \) and \( G_{12} = G_{13} = G_{23} = E_2/(2+2v) \). Here \( E_1 \) and \( E_2 \) are the principal compliance in the \( x_1 \) and \( x_2 \) directions, respectively. The degree of anisotropy is introduced through the parameter \( \beta \), and the material is isotropic when \( \beta = 1 \). \( \beta = 10 \sim 20 \) are typical for graphite epoxy composite materials. In the following presentation, the normalized stress intensity factors \( K^*/C_3 \) and the normalized energy release rate \( G^* \) versus \( \theta \) are plotted in Fig. 3 for \( \beta = 1.001 \). By letting \( \beta = 1.001 \) in the present formulation we can approximately model the behavior of isotropic materials. The results obtained by Obata et al. [23] are also shown in Fig. 3 with a circular symbol. It is seen from this figure that our results agree very well with them.

### 4.2. Accuracy validation

To validate the accuracy of the current numerical scheme, we consider the problem that the branched length is infinitesimally small for isotropic materials (\( l/c = 10^{-3} \)). Uniform shear load is applied at infinity and the condition of traction free on the crack faces is maintained when shear load is applied. The results of this problem for normalized stress intensity factors \( K_1^* \), \( K_{II}^* \) and normalized energy release rate \( G^* \) versus \( \theta \) are plotted in Fig. 3 for \( \beta = 1.001 \). By letting \( \beta = 1.001 \) in the present formulation we can approximately model the behavior of isotropic materials. The results obtained by Obata et al. [23] are also shown in Fig. 3 with a circular symbol. It is seen from this figure that our results agree very well with them.

### 4.3. Infinitesimal small branched length

We next consider again the problem of infinitesimally small branched length, but now the far field normal loading is compressive and the material is orthotropic. The ratio of applied normal compressive stress, \( \sigma_{22}^\infty \), to the applied shear stress \( \sigma_{21}^\infty \), is kept 0.5. Two cases are analyzed in the following, i.e., we select \( \mu = 0 \) and 0.3. This choice allows us to see the effect of \( \mu \) on the stress intensity factors. The results for \( K_1^* \) and \( K_{II}^* \) are shown in Fig. 4 for various branched angle \( \theta \) and anisotropic parameter \( \beta \). It is clear from these figures that even for the restricted choice of the present anisotropic parameter, the effect of \( \beta \) on stress intensity factors is significant. It can also be observed from Fig. 4...
that the existence of $\mu$ tends to reduce the magnitude of the stress intensity factors, as expected intuitively. Another observation is that $K_1$ increases initially with $h$ and then reaches maximum value at the position where the branched crack is approximately perpendicular to the main crack. Beyond that special position, $K_1$ however, will decrease with $h$. It is noted that at the place where maximum value of $K_1$ occurs the corresponding value for $K_{II}$ is approximately zero, as can be seen from Fig. 4(b).

4.4. Level curves for $k^*$ for various branched lengths and branched angles

It is worth investigating the behavior of $K_1^*(K_{II}^*)$ and $G^*$ for various branched lengths and branched angles. This investigation is helpful in the prediction of the direction of the quasistatic crack propagation whenever an appropriate fracture criterion, such as the maximum energy release rate, is used. Figs. 5–7 are the level curves for $k^*$ for different values of $\beta$. To plot these level curves, the increment of branched length is taken as 0.05 while the increment of the branched angle is taken as $3^\circ$ in the numerical computations. The branched angle ranges from $0^\circ$ to $90^\circ$, and the normalized branched length, i.e., $l/c$ is from 0 to 1. For simplicity, $\mu$ is taken to be zero which means that the surface of the main crack is now frictionless, although it maintains the ability to slide in the $x_1$-direction.

4.4.1. Level curves for $K_1^*$

We first note that for branched angles below about $45^\circ$, the level curves for $K_1^*$ (Figs. 5(a), 6(a) and 7(a)) are
curved lines when \( \ell/c \) is small; however, as the branched length grows longer (e.g., \( \ell/c > 0.1 \)) these level curves are approximately straight lines. Furthermore, when the length of the branched crack grows long enough, an interesting phenomenon can be observed for \( K_I^* \), i.e., the level curves are not only nearly straight lines but also are nearly horizontal. This implies that \( K_I^* \) will remain the same value even though the branched crack grows further. Another significant aspect is that there is a very flat region for \( K_{II}^* \), when \( \theta \) is approximately above 50°. This flat region is much wider and longer for the case \( \beta = 1.001 \) (nearly isotropic material) as compared to those for \( \beta = 10 \) and 20. The flat region implies that the stress intensity factor of \( K_I^* \) is nearly insensitive to the parameters \( \ell/c \) and \( \theta \) in that region.

### 4.4.2. Level curves for \( K_{II}^* \)

The behavior of those level curves for \( K_{II}^* \) are quite different from those for \( K_I^* \), as shown in Figs. 5(b), 6(b) and 7(b). For a nearly isotropic material (\( \beta = 1.001 \)), the level curves are nearly straight lines for \( \theta \) above approximately 40° (Fig. 5(b)). For other values of \( \beta \), the corresponding angle \( \theta \) for which level curves are to be approximately by straight lines is higher than that for isotropic material. Another observation is that for small branched angles \( K_{II}^* \) is more sensitive to the branched length \( \ell/c \).

### 4.5. Level curves for energy release rate for various branched lengths and branched angles

The level curves for \( G^* \) for various branched lengths and branched angles are shown in Figs. 8–10 for \( \beta = 1.001, 10 \) and 20, respectively. It is noteworthy that these level curves are similar to those for a saddle. The mountain regions are situated at the left-upper and right-lower corners in each figure. There is a saddle point in each figure. A flat region near the saddle point implies that the normalized energy release rate \( G^* \) will remain almost constant in that region. For instance, for \( \beta = 1.001, G^* \approx 1.3 \) for \( 0.1 < \ell/c < 0.4 \) and \( 45^\circ < \theta < 75^\circ \). One of the usefulness of these figures is to investigate the stability of the quasistatic crack propagation for the branched crack if the fracture toughness of a brittle material is known. For example, assuming that the normalized fracture toughness \( G_c^* \) of a given material is known, e.g. let \( G_c^* = 0.56 \) for \( \beta = 10 \) (Fig. 9). Assuming further that the branched crack starts to branch initially at angle \( \theta = 15^\circ \), and also assuming that this direction will be maintained when the branched crack grows, then the

![Fig. 8. Level curves for \( G^* \) for \( \beta = 1.001 \).](image-url)
branched crack will grow stably into the material until the branched crack length ($l/c$) reaches approximately 0.6 (see the dashed line in Fig. 9). If the branched crack grows further, i.e., $l/c > 0.6$, then the subsequent crack propagation will become unstable since $G^*$ is always greater than $G_c^*$ for $l/c > 0.6$ and $\theta = 15^\circ$. For example, considering the branched crack that starts to branch initially at an angle larger than approximately 30° (e.g., $\theta = 30^\circ$), then from Fig. 9 one would find that the branched crack will always grow unstably since $G^*$ is always greater than $G_c^*$ for $\theta = 30^\circ$ for all $l/c$.

**Appendix A**

The explicit form of the function $f'_h(z)$ is

$$
I = \sum_{i=1}^{3} \left\{ \text{diag} \left( F(z_1, z_1^0), F(z_2, z_2^0), F(z_3, z_3^0) \right) \right. 
\times B^{-1} R B I, B^\top \left( \frac{b^*}{2\pi} \right) - \text{diag} \left( F(z_1, z_1^0), F(z_2, z_2^0), F(z_3, z_3^0) \right) \right. 
\left. \times B^{-1} R B I, B^\top \left( \frac{b^*}{2\pi} \right) \right\}
$$

(A.1)

where

$$
F(z_k, z_k^0) = \frac{z_k^0}{z_k^3 - z_k^0} \left[ 1 - \frac{1}{z_k} \chi(z_k) \right]
$$

(A.2)

and matrices $I_i$ ($i = 1, 2, 3$) are defined by $I_1 = \text{diag}(1, 0, 0)$, $I_2 = \text{diag}(0, 1, 0)$, $I_3 = \text{diag}(0, 0, 1)$, respectively, while the explicit form of the function $f'_h(z)$ is

$$
I = \sum_{i=1}^{3} \left\{ \text{diag} \left( F(z_1, z_1^0), F(z_2, z_2^0), F(z_3, z_3^0) \right) \right. 
\times B^{-1} R B I, B^\top \left( \frac{b^*}{2\pi} \right) - \text{diag} \left( F(z_1, z_1^0), F(z_2, z_2^0), F(z_3, z_3^0) \right) \right. 
\left. \times B^{-1} R B I, B^\top \left( \frac{b^*}{2\pi} \right) \right\}
$$

(A.4)

As to the kernel $K(\xi, \eta)$ appearing in Eq. (28), the explicit expression of this kernel is

$$
K(\xi, \eta) = 2 \text{Im} \left\{ B \text{diag} \left( \frac{z_1^0}{z_1^3 + z_1^0}, \frac{z_2^0}{z_2^3 + z_2^0}, \frac{z_3^0}{z_3^3 + z_3^0} \right) B^\top \right. 
+ \sum_{j=1}^{3} \left\{ B \text{diag} \left( z_j^0 F(z_1, z_1^0), z_j^0 F(z_2, z_2^0), z_j^0 F(z_3, z_3^0) \right) \right. 
\times B^{-1} R B I, B^\top - B \text{diag} \left( z_j^0 F(z_1, z_1^0), z_j^0 F(z_2, z_2^0), z_j^0 F(z_3, z_3^0) \right) \right. 
\left. \times B^{-1} R B I, B^\top \right\}
$$

(A.5)

and the vector function $g(\xi)$ is

$$
g(\xi) = \text{Re} \left\{ B \text{diag} \left( z_1^0, z_2^0, z_3^0 \right) - 1 \right) B^{-1} \left( \begin{array}{c} r_1^\xi \cos(\theta) - r_1^\xi \sin(\theta) \\
\end{array} \right)
$$

(A.6)

In Eqs. (A.5) and (A.6), $z_1^\xi = \cos(\theta) + p_\xi \sin(\theta)$ and $z_2(\xi) = c + \xi z_2(t) (0 < \xi < t)$, $z_3(t) = c + z_3(t) (0 < t < \xi)$.

**References**