Optimum Operation Control for Multiple Machining Projects under Fixed Production Interval

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Abstract

The dynamic MRR control is the goal to optimize machining operations in the CNC machining systems. The concentration on the minimum-cost machining control with material removal rate (MRR) grows up in the computer-based machining industry. In this paper, the mathematical modeling with dynamic MRR control to minimize the machining cost as well as schedule the due-dates for multiple machining projects within a given production period is established. The optimum solution in manipulating the dynamic optimization task as well as the completeness and optimality of the solution are accomplished through Calculus of Variations. Moreover, the decision criteria for selecting the dynamic solution and the sensitivity analyses for key variables in the optimal solution are fully discussed. This study exactly explores the very promising solution to not only dynamically motivate the MRR in minimizing the machining cost, but also comprehensively schedule the due-dates for multiple machining quantities within a deterministic production period for both machine tool manufacturers and operations researchers.

Keywords: material removal rate; Calculus of Variations; due-date scheduling; multiple projects; dynamic optimization

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1. Introduction

It is mentioned that the dependency of a reliability model on the cutting conditions is the aim to optimize the manufacturing system [5]. Kim et al. [8] have discussed several time series modeling on the control of machining process for decision-making; nevertheless, none is characterized to complete the minimum machining cost, which is the major concern challenging the industry.

Our previous works [11, 16] have indicated that the material removal rate (MRR) is an important control factor of machining operation and the control of machining rate is also critical for production planners. Many researches on the dynamic MRR control were established in adaptive controllers for optimization of machining operations, a few of which are presented in this paper. Fuh et al. [4] have designed a variable structure system (VSS) controller on commercial computer numerical controlled (CNC) turning machines. It utilized overriding the spindle speed to dynamically manage the MRR. These PC-based controllers have also been realized to on-line override the programmed feed rates by Rober and Shin [13] on the CNC milling machines as well as by Kim and Kim [9] on the machining centers. Therefore, by overriding the feed rate and/or spindle speed on various CNC machines, the practicality of dynamically controlling the MRR for most CNC machining operations is doubtless.

The cost to machine each part is a function of the machining time [6]. While Kamien and Schwartz [7] described the marginal operation cost of production is a linear function of production rate, the operational cost of a machine is also proposed in direct proportion to the square of the material removal rate as the previous researches [11, 16] in this study. It is that the more machining rate results more
operational cost such as machine maintenance and depreciation costs. Besides, as the tool cost for a given amount of work was introduced into many of the machining optimization processes [1, 2], the overall tool cost for the machining projects is considered fixed in this study.

In addition, the promised due-date is to be determined during sales negotiations with the customers in many practical scheduling environments [12], and meeting the production deadline is the most desirable objective of management [14]. Kingsman et al. [10] have also stated that supplying products in response to a customer order in competition with other companies is the major problem confronting manufacturing. This is often complicated for multi-projects in a deterministic production period [10]. Therefore, the determination of due-dates for multiple machining projects within a production period is also critical to management in the industry.

As the advanced CNC machines are broadly used to perform from job shops to flexible manufacturing systems (FMS) [15], the attention in the minimum-cost machining control grows up in the systems. To minimize the production cost, the modeling of MRR control and the dynamic optimization for multiple machining quantities with single-tool operation are treated in this paper. The mathematical model proposed in this study definitely provides the functional solution to the technique, and contributes the economic approach to control the machining projects in a production period with profound insight.

2. Notations

The mathematical model in this paper is developed on the basis of the following notations.
2.1 Decision Variables

\( T_i \) = due-date for the \( i \)-th machining project, where \( i = 1,2,3, \ldots, n \).

\( M(t) \) = cumulated material machined during time interval \([0,t]\).

\( M'(t) \) = MRR at time \( t \).

2.2 Parameters

\([0,T_n]\) = controlling time interval of the production period.

\( A \) = lower limit of MRR.

\( a \) = average volume of material machined per unit part.

\( B \) = upper limit of MRR.

\( bM'(t) \) = marginal operation cost at the material removal rate \( bM'(t) \); where \( b \) is a constant.

\( bM''(t) \) = operational cost at time \( t \).

\( c \) = overall holding cost of unit chip per unit time, where \( c = h_1 + \frac{h_2}{a} \).

\( c_l \) = labor cost per unit time.

\( h_1 \) = chip holding cost of unit chip per unit time.

\( h_2 \) = product holding cost of unit finished part per unit time.

\( n \) = number of machining projects to be scheduled within the production period \( \bar{T} \).

\( Q_j \) = machining quantity of the \( j \)-th project, where \( j = 1,2,3, \ldots, n \).

\( \bar{T} \) = deterministic production period for the \( n \) machining projects, which is given by the company.
3. Mathematical Modeling

In this paper, the cutting process is regarded to be a continuous single-tool turning operation without breakdown as in our previous works [11, 16]. Within the production period, the multiple machining projects are scheduled on the basis of FCFS (first come first serve), which is commonly practiced in the industry. The upper limit and lower limit of MRR are generated from the maximum and minimum acceptable conditions (speed, feed rate and depth of cut) suggested in the machining handbook, respectively. Therefore, no chattering or scrapping of parts will occur during machining.

The machining costs are divided into two different categories, the operational cost and holding cost [11, 16]. Whilst the operational cost of the machine is directly proportional to the square of the MRR [11, 16] and the labor cost per unit time is considered to be fixed [11], \( \int_{0}^{T_n} bM^2(t)dt \) and \( \int_{0}^{T_n} c_i dt \) signify the operational cost of the machine and the labor cost during time interval \([0, T_n]\) respectively in this study.

As many of the real manufacturing cases, all chips from cutting and the finished products are usually held and stored at the machine shop until the machining projects are done. Therefore, \( \int_{0}^{T_n} h_1 M(t)dt \) denotes the chip holding cost during time interval \([0, T_n]\). To practically compensate the variation between project due-dates and order deliveries, the maximum product holding cost \( \int_{0}^{T_n} h_2 \frac{M(t)}{a} dt \) during time interval \([0, T_n]\) is introduced. In addition, the time for tool changes is negligible comparing with the production period.
With all the viewpoints above, the minimum-cost objective function is thus composed and constructed as below.

\[
\text{minimize}_{M,T} \left\{ \int_0^{T_n} \left[ bM^{(2)}(t) + h_1 M(t) + h_2 \frac{M(t)}{a} + c_i \right] dt \right\}
\]

Noted that \( c = h_1 + \frac{h_2}{a} \) is the overall holding cost of unit chip per unit time.

Rearranging the objective function above, the mathematical model and its constraints are then elaborated as below.

\[
\begin{align*}
\text{minimize}_{M,T} & \left\{ \int_0^{T_n} \left[ bM^{(2)}(t) + cM(t) + c_i \right] dt \right\} \\
\text{subject to} & \quad M(0) = 0 \\
& \quad M(T_i) = a \sum_{j=1}^{Q_i} i = 1,2,3,\ldots,n \\
& \quad T_n = \bar{T} \\
& \quad A \leq M'(t) \leq B, \quad \forall t \in [0,T_n]
\end{align*}
\]

4. Optimal Solution

In this study, \( M'(t) \) is time-continuous and differentiable. Let \( M'(t) \) and \( T_i^* \) be the optimal solution of the proposed model, and suppose that time interval \( (0,\bar{T}) \) is the maximal subinterval of \([0,\bar{T}]\) to satisfy Euler Equation [3, 7]. There are two feasible cases to be discussed as follows.

4.1 Situation 1: \( M'(t) \) will not touch \( B \) before the production period \( \bar{T} \). (\( \bar{T} = \bar{T} \))

The optimal solution for Situation 1 is shown as follows:
\[ M^*'(t) = \frac{c}{2b}t + \frac{a \sum_{j=1}^{n} Q_j}{T} - \frac{cT}{4b} \quad \forall t \in [0, \bar{T}] \] (1)

\[ M^*(t) = \frac{c}{4b} t^2 + \left( \frac{a \sum_{j=1}^{n} Q_j}{\bar{T}} - \frac{cT}{4b} \right) t \quad \forall t \in [0, \bar{T}] \] (2)

The detailed procedures are described in Appendix A.

**PROPERTY:** If the line \( y = M^*'(t) \) touches the line \( y = B \), these two lines should overlap to be \( y = B \) from the touch point \( \tilde{t} \) to the end point \( T_n \).

**Proof:** From Eq.(1), \( y = M^*'(t) \) is a strictly increasing linear function of \( t \). And it holds for the subinterval of \( [0, \tilde{t}] \) subject to \( A \leq M^*'(t) \leq B \). Since the straight line in the time interval \( [\tilde{t}, T_n] \) (shown in Figure 1) cannot exist because it contradicts Euler Equation [3, 7] to be a decreasing linear function of \( t \), the property is verified.

**4.2 Situation 2:** \( M^*'(t) \) will touch \( B \) before the production period \( \bar{T} \). \((\tilde{t} \in [0, \bar{T}])\)

We assume that the material removal rate \( M^*'(t) \) will reach the upper limit \( B \) at \( t = \tilde{t} \) where \( \tilde{t} \in [0, \bar{T}] \). The optimal solution for **Situation 2** is shown as follows:

\[ \tilde{t} = \bar{T} - \frac{\sum_{j=1}^{n} Q_j}{B} + \frac{(bB^2 - c_j)}{cB} \] (3)

\[ M^*'(t) = \begin{cases} 
\frac{c}{2b}t + \frac{a \sum_{j=1}^{n} Q_j}{\bar{T}} - \frac{c\bar{T}}{4b} & \text{if } t \in [0, \tilde{t}] \\
B & \text{if } t \in (\tilde{t}, \bar{T}] 
\end{cases} \] (4)
\[
M^*(t) = \begin{cases}
\frac{c}{4b}t^2 + \left( a \frac{\sum_{j=1}^{n} Q_j}{\bar{T}} - \frac{c\bar{T}}{4b} \right) t & \text{if } t \in [0, \bar{t}]

M^*(\bar{t}) + B(t - \bar{t}) & \text{if } t \in (\bar{t}, \bar{T}]
\end{cases}
\] (5)

The detailed procedures are described in Appendix B.

The algorithm in achieving the optimal solution of the proposed model provides a continuous function indicating the optimal path to be followed by the variables through time or space. By solving the equation, \( M^*(T_i) = a \sum_{j=1}^{i} Q_j \), at \( i = 1, 2, 3, \ldots, n - 1 \); the due-dates for other projects can then be found. Thus, the due-date scheduling for multiple machining projects can then be further approached.

It is also argued [3, 11, 16] that using the properties of the Calculus of Variations for dynamic optimization, the completeness and the optimality of the solution are absolutely guaranteed.

5. Decision Criteria

The decision criteria for selecting the optimal solution in the two feasible cases are to be discussed as follows.

From Eq.(3), two decision criteria are found and described as follows:

When \( \frac{a \sum_{j=1}^{n} Q_j}{B} \leq \frac{(bB^2 - c_i)}{cB} \), it means \( \tilde{t} \geq \bar{T} \). This contradicts the assumption of Situation 2. It is that the optimal control of the material removal rate \( M^*(t) \) will not touch the upper MRR limit within the production period \( \bar{T} \). The optimal solution
is Situation 1.

\[ \frac{a \sum_{j=1}^{n} Q_j}{B} > \frac{(bB^2 - c_j)}{cB} , \]

it means \( \bar{\tau} < \bar{T} \). This is that the optimal control of the material removal rate \( M^*(t) \) will touch the upper MRR limit within the production period \( \bar{T} \). The optimal solution is Situation 2.

6. Sensitivity Analysis

6.1 Sensitivity Analysis for Situation 1.

From Eq. (1) and (2), it is found that both the cumulative volume of material machined \( M^*(t) \) and the optimum material removal rate \( M^*(t) \) are increasing with the material removal per unit product \( a \) or the machining quantity of a project \( Q_j \). It is also shown that increasing the production period \( \bar{T} \) will decrease both the cumulative volume of material machined \( M^*(t) \) and the optimum material removal rate \( M^*(t) \).

The overall sensitivity analysis for Situation 1 is shown in Table 1.

6.2 Sensitivity Analysis for Situation 2.

From Eq. (3), it is obtained that the time \( \bar{\tau} \) to reach upper MRR limit is in inverse proportion to the material removal per unit product \( a \), the marginal operation constant \( b \), the machining quantity of a project \( Q_j \) or the labor cost per unit time \( c_j \). It is also noted that increasing the overall holding cost of unit chip per unit time \( c \) or the production period \( \bar{T} \) will delay the time \( \bar{\tau} \) to reach the upper limit.
By Eq. (3) and (5), the cumulative volume of material machined \( M'(t) \) is increasing with the material removal per unit product \( a \) or the machining quantity of a project \( Q_j \).

The overall sensitivity analysis for *Situation 2* is shown in Table 2.

## 7. Summary and Remarks

The operational cost, holding costs, average material machined per unit part, multiple machining quantities, labor cost per unit time, unconstrained machining due-date, and upper/lower limit of \( MRR \) are integrated concurrently to investigate the optimal dynamic control of \( MRR \) as well as to determine the production due-dates for multiple machining quantities with single-tool operation. This is deemed complicated; however, the problem becomes stabilized through this study.

The \( MRR \) is an important control issue of the machining operation, and the control of machining rate is also critical for production planners in modern *CNC* machining industry. There are two characters of the optimal \( MRR \) control derived from this study. First, the optimal \( MRR \) control \( M'(t) \) is a strictly increasing linear function of \( t \) before reaching the upper \( MRR \) limit. Second, by the PROPERTY described before, if the material removal rate \( M'(t) \) touches the upper limit \( B \), the optimal \( MRR \) will settle to be the upper limit \( B \) for the rest of the production period.

Additionally, the decision criteria for selecting the optimal control of \( MRR \), the sensitivity analyses on various key variables in the optimal solution, as well as the completeness and optimality of the solution are fully discussed. The practicality of dynamically \( MRR \) control for most *CNC* machining operations is doubtless. With the
optimal solution in this study, the machine maintenance, the manufacturing scheduling, machining cost estimating, and even the contract negotiation can then be further approached.

Minimizing the machining cost of multiple production projects in a certain manufacturing period has become essential for computer-based machining industry. In this paper, the modeling and optimization of the dynamic MRR control as well as the due-date scheduling of multiple machining quantities within a given production period are addressed. An experimental study on a typical Computer Numerical Control (CNC) machine with PC-based Digital Signal Processor (DSP) to on-line realize and simulate the optimal MRR control of machining projects is being conducted following this paper. Future researches with the modeling of dynamic optimization on various machining processes are encouraged in this study. This study not only contributes a better and practical conception of machine tool control to the techniques, but also generates a reliable and applicable approach to minimize the machining cost as well as to schedule the multiple projects in a deterministic production period.

**Appendix A:** The optimal solution for *Situation 1*.

Suppose that the material removal rate $M'(t)$ will not reach the upper limit $B$ before time $\bar{T}$. Also, let $F = bM'(t) + cM(t) + c_1$.

From Euler Equation [3,7], $F_M' = \frac{d}{dt}F_M'$, it is derived that

$$c = \frac{d}{dt}2bM'(t)$$
There exists a constant $\bar{k}_1$ to satisfy

$$M'(t) = \frac{c}{2b} t + \bar{k}_1 \quad \forall t \in [0, \bar{T}] \quad (A1)$$

Integrating Eq. (A1) with $t$, it is obtained that

$$M(t) = \frac{c}{4b} t^2 + \bar{k}_1 t + \bar{k}_2 \quad \forall t \in [0, \bar{T}] \quad (A2)$$

Introducing the boundary condition, $M(0) = 0$, into Eq. (A2); then

$$\bar{k}_2 = 0 \quad (A3)$$

Providing $M(\bar{T}) = a \sum_{j=1}^{n} Q_j$ and Eq. (A3) into Eq. (A2), we have

$$\bar{k}_1 = \frac{a \sum_{j=1}^{n} Q_j}{\bar{T}} - \frac{c \bar{T}}{4b} \quad (A4)$$

Substituting Eq. (A3) and (A4) into Eq. (A1) and (A2); $M'(t)$ and $M'(t)$ are then obtained.

**Appendix B:** The optimal solution for *Situation 2.*

Before $M'(t)$ touches the upper limit $B$, Eq. (1) and (2) are satisfied. Besides, when it reaches the upper limit $B$, the PROPERTY is then applied.

Using the transversality condition for free end point $\bar{t} \quad [3, 7]$, $F - M^' \mathcal{T}_{\bar{t}} = 0$;

it is derived that

$$-bM'(\bar{t}) + cM(\bar{t}) + c_i = 0 \quad (B1)$$

Introducing $M'(\bar{t}) = B$ into Eq. (B1), it is obtained

$$M(\bar{t}) = \frac{bB^2 - c_i}{c} \quad (B2)$$
With the PROPERTY and the boundary condition \( M(\bar{T}) = a \sum_{j=1}^{n} Q_j \), it is noted that

\[
a \sum_{j=1}^{n} Q_j = M(\bar{t}) + B(\bar{T} - \bar{t})
\]  

(B3)

Providing Eq.(B2) into Eq. (B3), we have

\[
\bar{T} - \bar{t} = \frac{a \sum_{j=1}^{n} Q_j}{B} - \frac{(bB^2 - c_j)}{cB}
\]  

(B4)

Rearranging Eq. (B4), the touch point \( \bar{t} \) is derived.

With the PROPERTY, the optimal solution, \( M^*(t) \) and \( M^*(t) \), is then obtained.

References


![Diagram](image)

**Figure 1.** Possible condition of line $y=B$ and $y=M^{'}(t)$
### Table 1. The sensitivity analysis for Situation 1. ($\tilde{T} = T$)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Decision Variables</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$Q_j$</th>
<th>$T$</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^*(t)$</td>
<td>$+$</td>
<td>#</td>
<td>#</td>
<td>$+$</td>
<td>$-$</td>
<td></td>
<td>Eq.(1)</td>
</tr>
<tr>
<td>$M^*(t)$</td>
<td>$+$</td>
<td>#</td>
<td>#</td>
<td>$+$</td>
<td>$-$</td>
<td></td>
<td>Eq.(2)</td>
</tr>
</tbody>
</table>

“$+$”: Decision variable is directly proportional to the parameter
“$-$”: Decision variable is inversely proportional to the parameter
“#”: Decision variable depends on the changes of other relevant parameters.

### Table 4. The sensitivity analysis for Situation 2. ($\tilde{T} \in [0, T]$)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Decision Variables</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$B$</th>
<th>$Q_j$</th>
<th>$T$</th>
<th>$c_i$</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{T}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
<td>#</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td></td>
<td>Eq.(3)</td>
</tr>
<tr>
<td>$M^*(t)$</td>
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<td>#</td>
<td>#</td>
<td>#</td>
<td>$+$</td>
<td>#</td>
<td>#</td>
<td></td>
<td>Eq.(3), (5)</td>
</tr>
</tbody>
</table>

“$+$”: Decision variable is directly proportional to the parameter
“$-$”: Decision variable is inversely proportional to the parameter
“#”: Decision variable depends on the changes of other relevant parameters.